DIMENSION AND MEASURE THEORY OF SELF-SIMILAR STRUCTURES WITH NO SEPARATION CONDITION

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Declaration

I, Ábel Farkas, hereby certify that this thesis, which is approximately 22000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2011 and as a candidate for the degree of Doctor of Philosophy in September 2011; the higher study for which this is a record was carried out in the University of St Andrews between 2011 and 2015.

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Abstract

We introduce methods to cope with self-similar sets when we do not assume any separation condition. For a self similar set $K \subseteq \mathbb{R}^d$ we establish a similarity dimension-like formula for Hausdorff dimension regardless of any separation condition. By the application of this result we deduce that the Hausdorff measure and Hausdorff content of K are equal, which implies that K is Ahlfors regular if and only if $\mathcal{H}^t(K) > 0$ where $t = \dim_H K$. We further show that if $t = \dim_H K < 1$ then $\mathcal{H}^t(K) > 0$ is also equivalent to the weak separation property. Regarding Hausdorff dimension, we give a dimension approximation method that provides a tool to generalise results on non-overlapping self-similar sets to overlapping self-similar sets.

We investigate how the Hausdorff dimension and measure of a self-similar set $K \subseteq \mathbb{R}^d$ behave under linear mappings. This depends on the nature of the group $\mathcal{T}$ generated by the orthogonal parts of the defining maps of K. We show that if $\mathcal{T}$ is finite then every linear image of K is a graph directed attractor and there exists at least one projection of K such that the dimension drops under projection. In general, with no restrictions on $\mathcal{T}$ we establish that $\mathcal{H}^t\left(L \circ O(K)\right) = \mathcal{H}^t\left(L(K)\right)$ for every element O of the closure of $\mathcal{T}$, where L is a linear map and $t = \dim_H K$. We also prove that for disjoint subsets A and B of K we have that $\mathcal{H}^t\left(L(A) \cap L(B)\right) = 0$. Hochman and Shmerkin showed that if $\mathcal{T}$ is dense in $SO(d, \mathbb{R})$ and the strong separation condition is satisfied then $\dim_H\left(g(K)\right) = \min\left\{\dim_H K, l\right\}$ for every continuously differentiable map g of rank l. We deduce the same result without any separation condition and we generalize a result of Eroğlu by obtaining that $\mathcal{H}^t(g(K)) = 0$.

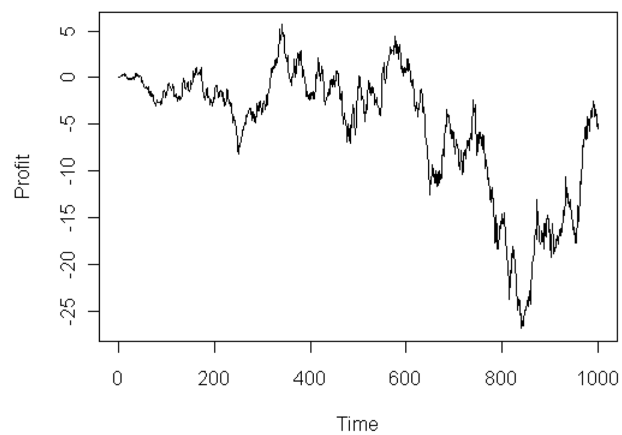
We show that for the attractor $(K_1, \ldots, K_q)$ of a graph directed iterated function system, for each $1 \leq j \leq q$ and $\varepsilon > 0$ there exists a self-similar set $K \subseteq K_j$ that satisfies the strong separation condition and $\dim_H K_j - \varepsilon < \dim_H K$. We show that we can further assume convenient conditions on the orthogonal parts and similarity ratios of the defining similarities of K. Using this property we obtain results on a range of topics including on dimensions of projections, intersections, distance sets and sums and products of sets.

We study the situations where the Hausdorff measure and Hausdorff content of a set are equal in the critical dimension. Our main result here shows that this equality holds for any subset of a set corresponding to a nontrivial cylinder of an irreducible subshift of finite type, and thus also for any self-similar or graph directed self-similar set, regardless of separation conditions. The main tool in the proof is an exhaustion lemma for Hausdorff measure based on the Vitali's Covering Theorem. We also give several examples showing that one cannot hope for the equality to hold in general if one moves in a number of the natural directions away from 'self-similar'. Finally we consider an analogous version of the problem for packing measure. In this case we need the strong separation condition and can only prove that the packing measure and δ-approximate packing pre-measure coincide for sufficiently small $\delta > 0$.

Contents

1 Introduction

Fractals are usually considered highly irregular objects, nevertheless, there is no exact definition of fractals. Imagine you hold a head of broccoli in your hand and you pick a little piece of it. If you imagine that it becomes bigger then it would look roughly similar to the original head of broccoli (see the pictuere below, self-similarity in romanesco broccoli [Photo courtesy PDPhoto.org, public domain at http://pdphoto.org/PictureDetail.php?mat=pdef&pg=8232]). That is one sort of typical fractal feature, that the object exhibits similar structure at every scale. This similarity does not have to be deterministic, it can be just similar in behaviour. For instance imagine the graph of a stock price. You would see a very irregular curve and if you just zoom into a small part of it than you would find a similarly irregular curve (see the picture below, graph of a Brownian motion [Photo courtesy Edwin Grappin at http://probaperception.blogspot.co.uk/2012/10/generate-stock-option-prices-how-to.html]).



Fractal Geometry develops tools to study these rough objects for which the classical tools of Geometry fail to apply. One of the most important notions of the field is the dimension of these fractal objects, which somehow tries to estimate how much the set fills up the space. If you consider a line segment and compare its length to a twice as long line segment's then the ratio of their length is obviously 2. If you consider a square and compare its area to a square's area with side length double as much then the ratio of their areas is $2 \cdot 2 = 2^2$, i.e. the ratio of their area is the square of the ratio of their side length. If you consider a cube and compare its volume to that of another cube of double that long diameter then the ratio of their volume is $2 \cdot 2 \cdot 2 = 2^3$, i.e. the cube of the ratio of their diameter. So the quantity in the power represents the dimension. Following this argument we try to capture the meaning of the dimension of an object by considering what is the growth rate of their 'measure' as you change the scale at which the object is viewed. This was introduced by Felix Hausdorff in 1918 [38]. Studying these notions one can find that in fact, the dimension is not necessarily a whole number, the power in the growth rate does not need to be an integer. Beniot B. Mandelbrot termed such objects ‘fractals’, from the Latin word ‘fractus’ meaninig broken. Mandelbrot popularised fractals by indicating their occurrence in nature [48, 49].

One of the most common examples of a fractal is the middle-third Cantor set (see Figure A). Take the unit interval $[0, 1]$, then remove the middle-third open interval $(1/3, 2/3)$, so we end up with two intervals of length $1/3$. Then we delete the middle third open interval of these intervals to gain four intervals of length $1/9$. We continue this process infinitely many times and by the Cantor axiom the remaining set is not empty but is what we call the middle-third Cantor set. Observe that the middle-third Cantor set is built up as the union of two disjoint scaled down copies of itself and the scaling ratio is $1/3$. Hence by using the growth rate property of the dimension the ‘measure’ μ of the middle-third Cantor set is two times the ‘measure’ of its scaled down copy, i.e. $\mu = 2 \cdot (1/3)^s \cdot \mu$ where s is the dimension of the middle-third Cantor set. Thus by solving the equation we get that $s = \log 2/ \log 3$, so the dimension of the middle-third Cantor set is not a whole number. When a set is built up as the union of the scaled down copies of itself then we call such a set a ‘self-similar’ set.

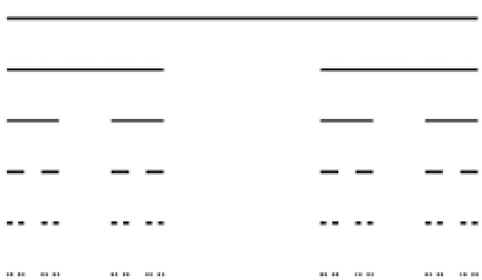


Figure A.

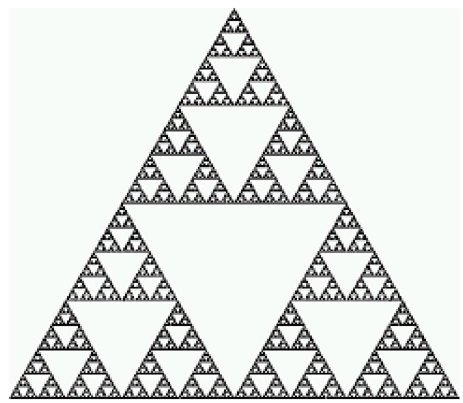


Figure B.

Another well-known example of a self-similar set is the Sierpinski triangle (see Figure B) which we construct by taking an equilateral triangle and subdividing into four triangles and deleting the middle triangle, then repeating the process with all the triangles again and again in infinitely many steps. The Sierpinski triangle is the union of three scaled down copies of itself and the scaling ratio is $1/2$. However, in this case the scaled down copies are not completely disjoint. The corners of scaled down triangles touch. On the other hand, this is only a mild overlapping because the overlap is a single point for each pair of triangles. So the ‘measure’ of these mild overlaps is zero, thus the ‘measure’ of the Sierpinski triangle still can be expressed as the sum of the ‘measure’ of the scaled down pieces. So we can again calculate the dimension: $\mu = 3 \cdot (1/2)^s \cdot \mu$, thus the dimension is $s = \log 3/ \log 2$. The reason that the overlaps are not significant is that the scaled down pieces are ‘nicely separated’. In these ‘nicely separated’ situations analysis of the fractals is much simpler but when we have no ‘nice separation’ then the study of these sets becomes notoriously difficult. In this work our aim is to develop tools and methods treating the general case when we do not assume ‘nice separation’.

This study is structured as follows. Chapter 2 contains three main sections. Section 2.1 introduces the formal definitions of dimension and summarises their important properties. In Section 2.2 we define the main families of set that we investigate in this study. The three main classes are ‘self-similar sets’, ‘attractors of graph directed iterated function systems’ and ‘subshifts of finite type’. In Section 2.3 we collect some important properties of orthogonal transformations. It turns out that many geometric properties of the self-similar structures depend on the orthogonal part of their defining maps. Chapter 3 provides methods and tools to treat the lack of separation conditions in the proofs. We draw a few interesting conclusions as a consequence of these methods including the equality of Hausdorff content and Hausdorff measure for self-similar sets and an interesting characterisation of the ‘weak separation property’ for sets of Hausdorff dimension strictly less than 1. Most of this work was done in [25, 26]. In Chapter 4 we study how self-similar sets behave under projections and linear mappings. In particular, we investigate the Hausdorff dimension and Hausdorff measure of these projections. This chapter contains work from [25]. Chapter 5 gives us tools to generalise results about self-similar sets to attractors of graph directed iterated function systems and subshifts of finite type. This material can be found in [24]. In Chapter 6 we discuss the situations where the Hausdorff content and the Hausdorff measure agree for subshifts of finite type and for graph directed iterated function systems. We study an analogous question for packing measure. These are joint results with Fraser [26].

2 Preliminary definitions and results

In this chapter we introduce the basic notions of fractal geometry such as dimensions and measures and establish the notation that we shall use. Then we define the fractal objects that we study. The objects we investigate all exhibit some sort of self-similarity. Finally, we state some useful results about orthogonal transformations that we use in the latter chapters.

2.1 Dimensions and measures

In this section we define the most common notions of measures and dimensions. These quantities determine the size of a set in the sense that how much it fills up the space. We also summarise their most important basic properties.

2.1.1 Hausdorff dimension

Hausdorff measure and dimension were introduced by Hausdorff [38]. Let $H \subseteq \mathbb{R}^d$, $s \in [0, \infty)$ and $\delta > 0$, then the *s-dimensional δ-approximate Hausdorff pre-measure of H* is

$$\mathcal{H}^s_\delta(H) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam}(H_n)^s : H \subseteq \bigcup_{n=1}^{\infty} H_n, \, \operatorname{diam}(H_n) \leq \delta \right\},$$

the *s-dimensional Hausdorff content of H* is

$$\mathcal{H}^s_\infty(H) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam}(H_n)^s : H \subseteq \bigcup_{n=1}^{\infty} H_n \right\}$$

and the *s-dimensional Hausdorff outer measure of H* is

$$\mathcal{H}^s(H) = \sup_{\delta > 0} \mathcal{H}^s_\delta(H) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(H).$$

It follows from the definitions that for every $\delta > 0$

$$\mathcal{H}^s_\infty(H) \leq \mathcal{H}^s_\delta(H) \leq \mathcal{H}^s(H). \tag{2.1}$$

In case of zero measure we have equality, $\mathcal{H}^s_\infty(H) = 0$ if and only if $\mathcal{H}^s(H) = 0$. We say that an $M \subseteq \mathbb{R}^d$ is *$\mathcal{H}^s$-measurable* if

$$\mathcal{H}^s(H) = \mathcal{H}^s(H \cap M) + \mathcal{H}^s(H \setminus M) \tag{2.2}$$

for every $H \subseteq \mathbb{R}^d$ and we say that the *Hausdorff measure of M* is $\mathcal{H}^s(M)$. Let $\mathcal{M}(\mathcal{H}^s)$ denote the set of all $\mathcal{H}^s$-measurable sets of $\mathbb{R}^d$, let $\mathcal{B}(\mathbb{R}^d)$ be the set of all Borel sets of $\mathbb{R}^d$ and $\mathcal{P}(\mathbb{R}^d)$ be the set of all subsets $\mathbb{R}^d$.

Theorem 2.1. *We have that $\mathcal{M}(\mathcal{H}^s)$ is a σ-algebra, $\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{M}(\mathcal{H}^s) \subseteq \mathcal{P}(\mathbb{R}^d)$, $\mathcal{H}^s$ is an outer measure on $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{H}^s$ is a measure on $\mathcal{M}(\mathcal{H}^s)$.*

For the proof see [52, Theorem 4.2] and [68, Theorem 3].

Lemma 2.2. *Let $H \subseteq \mathbb{R}^d$ be such that $\mathcal{H}^s_\infty(H) = \mathcal{H}^s(H) < \infty$ where $s = \dim_{\mathrm{H}} F$. Then for every $\mathcal{H}^s$-measurable subset $A \subseteq H$ we also have $\mathcal{H}^s_\infty(A) = \mathcal{H}^s(A)$.*

Proof. It follows by (2.2) that

$$\mathcal{H}^s(H) = \mathcal{H}^s(A) + \mathcal{H}^s(H \setminus A)$$

thus using (2.1) it follows that

$$\mathcal{H}^s(H) = \mathcal{H}^s(A) + \mathcal{H}^s(H \setminus A) \geq \mathcal{H}^s_\infty(A) + \mathcal{H}^s_\infty(H \setminus A) \geq \mathcal{H}^s_\infty(H) = \mathcal{H}^s(H)$$

so we must have $\mathcal{H}^s(A) = \mathcal{H}^s_\infty(A)$. □

For an arbitrary set $A \subseteq \mathbb{R}^d$ a $\mathcal{H}^s$-measurable set B such that $A \subseteq B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$ is called a *$\mathcal{H}^s$-measurable hull of A*.

Lemma 2.3. *For every set $A \subseteq \mathbb{R}^d$ there exists a Borel set B that is a $\mathcal{H}^s$-measurable hull of A.*

From [52, Theorem 4.4] it can be deduced that there exists a $\mathcal{H}^s$-measurable hull of A that is a G_δ set. So there exists a Borel set B that is a $\mathcal{H}^s$-measurable hull of A.

Lemma 2.4. *For a set $A \subseteq \mathbb{R}^d$ such that $\mathcal{H}^s(A) < \infty$, let B be a $\mathcal{H}^s$-measurable hull of A and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. Then $\mathcal{H}^s\left(L(A)\right) = \mathcal{H}^s\left(L(B)\right)$.*

Proof. Let C be a $\mathcal{H}^s$-measurable hull of $L(A)$ such that $C \subseteq L(B)$. Then $B \cap L^{-1}(C)$ is also a $\mathcal{H}^s$-measurable hull of A. It follows that $\mathcal{H}^s(B \cap L^{-1}(C)) = \mathcal{H}^s(A) = \mathcal{H}^s(B)$, thus $\mathcal{H}^s\left(B \setminus B \cap L^{-1}(C)\right) = 0$. Hence $\mathcal{H}^s\left(L(B) \setminus C\right) = 0$ and so $\mathcal{H}^s\left(L(A)\right) = \mathcal{H}^s\left(L(B)\right)$. □

Theorem 2.5. *Let $H \in \mathcal{M}(\mathcal{H}^s)$ such that $\mathcal{H}^s(H) < \infty$. Then there exists an increasing sequence of compact sets $K_n \subseteq H$, $(n \in \mathbb{N})$ such that $\lim_{n\to\infty} \mathcal{H}^s(K_n) = \mathcal{H}^s(H)$.*

For the proof see the discussion after [52, Corollary 4.5].

It is easy to see that $\mathcal{H}^0$ is the counting measure. Let $\mathcal{L}^d$ denote the *d-dimensional Lebesgue measure*. The following theorem shows that the d-dimensional Hausdorff measure in $\mathbb{R}^d$ is the same as the d-dimensional Lebesgue measure up to a scalar multiple.

Theorem 2.6. *There exists $0 < c_d < \infty$ such that $\mathcal{H}^d(H) = c_d \cdot \mathcal{L}^d(H)$ for every $H \subseteq \mathbb{R}^d$.*

For the proof see [52, Theorem 3.4].

Theorem 2.7. *For every $A \subseteq \mathbb{R}^d$ there exists a unique $s \in [0, d]$ such that*
i) $t < s \Rightarrow \mathcal{H}^t(A) = \infty$
ii) $s < t \Rightarrow \mathcal{H}^t(A) = 0$.

For the proof see [52, Theorem 4.7].

Notation 2.8. We call this unique s the *Hausdorff dimension of* A and we denote it by $\dim_H A$.

We call a set A an s-*set* if $0 < \mathcal{H}^s(A) < \infty$. Often an s-set is assumed to be Borel measurable but we do not require it to be. Then of course $\dim_H(A) = s$.

Proposition 2.9. *Let $A \subseteq B \subseteq \mathbb{R}^d$, then $\dim_H(A) \leq \dim_H(B)$.*

Proposition 2.10. *Let $A_n \subseteq \mathbb{R}^d$ be a sequence of sets ($n \in \mathbb{N}$), then $\dim_H \bigcup_{n\in\mathbb{N}} A_n = \sup_{n\in\mathbb{N}} \dim_H A_n$.*

Proposition 2.9 and Proposition 2.10 follow easily from the definition of Hausdorff dimension.

Proposition 2.11. *Let $H \subseteq \mathbb{R}^d$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ ($d_2 \in \mathbb{N}$) be a Lipschitz map with Lipschitz constant c. Then $\mathcal{H}^s(f(H)) \leq c^s \cdot \mathcal{H}^s(H)$ and $\dim_H(f(H)) \leq \dim_H(H)$.*

For the proof see [14, Proposition 2.2, Corollary 2.4]. The following corollary is immediate.

Corollary 2.12. *Let $H \subseteq \mathbb{R}^d$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ ($d_2 \in \mathbb{N}$) be a bi-Lipschitz map. Then*
i) $\dim_H(f(H)) = \dim_H(H)$,
ii) $\mathcal{H}^s(f(H)) = 0$ if and only if $\mathcal{H}^s(H) = 0$,
iii) $\mathcal{H}^s(f(H)) = \infty$ if and only if $\mathcal{H}^s(H) = \infty$.

Let $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. We define the *Euclidean operator norm of* L as

$$\|L\| = \sup_{x\in\mathbb{R}^d, \|x\|=1} \|Lx\|$$

where $\|y\|$ denotes the Euclidean norm of a vector y. Every linear map $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ is a Lipschitz map with Lipschitz constant $\|L\|$. We state the following well-known lemma, which we use without reference throughout this study.

Lemma 2.13. *Let $H \subseteq \mathbb{R}^d$ and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. Then $\mathcal{H}^s(L(H)) \leq \|L\|^s \cdot \mathcal{H}^s(H)$ and $\dim_H(L(H)) \leq \dim_H(H)$.*

Lemma 2.13 follows from Proposition 2.11.

A *similarity transformation of* $\mathbb{R}^d$ is a map $S : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\|S(x) - S(y)\| = r \cdot \|x - y\|$ for every $x, y \in \mathbb{R}$ where $r \in (0, \infty)$ is called the *similarity ratio of* S. Every similarity can be uniquely written in the form $S(x) = r \cdot T(x) + v$ where r is the similarity ratio, T is an orthogonal transformation and $v \in \mathbb{R}^d$ is a translation vector. If $r \in (0, 1)$ then we say that S is a *contracting similarity*. Two sets $A, B \subseteq \mathbb{R}^d$ are *similar* if there exists a similarity S such that $S(A) = B$.

The next lemma easily follows from the definition of the Hausdorff measure.

Lemma 2.14. *Let $A \subseteq \mathbb{R}^d$ and S be a similarity of $\mathbb{R}^d$, with similarity ratio r. Then $\mathcal{H}^s(S(A)) = r^s \cdot \mathcal{H}^s(A)$ for every $s \in [0, \infty)$ and hence $\dim_H(S(A)) = \dim_H(A)$.*

2.1.2 Vitali's covering theorem

Let $H \subset \mathbb{R}^d$. A collection of sets $\mathcal{A}$ is called a *Vitali cover of* H if for each $x \in H$, $\delta > 0$, there exists $A \in \mathcal{A}$ with $x \in A$ and $0 < \operatorname{diam}(A) < \delta$. An outer measure μ on $\mathbb{R}^d$ is called a Borel measure if μ is a measure on $\mathcal{B}(\mathbb{R}^d)$. We say that μ is a *Radon measure on* $\mathbb{R}^d$ if μ is a locally finite Borel measure and for every set $A \subseteq \mathbb{R}^d$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$. The following theorem is Vitali's covering theorem [52, Theorem 2.8].

Theorem 2.15. *Let μ be a Radon measure on $\mathbb{R}^d$, $H \subset \mathbb{R}^d$ be a Borel set and $\mathcal{A}$ be a family of closed balls of $\mathbb{R}^d$ such that for every $x \in H$ and $r > 0$ there is $B \in \mathcal{A}$ that is centred at x and has radius less than r. Then there exists a disjoint sequence of balls (finite or countable) $B_1, B_2, \ldots \in \mathcal{A}$ such that $\mu\left(H \setminus \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = 0$.*

It follows from Lemma 2.3 that if $\mathcal{H}^s|_A$ is a locally finite measure for a set $A \subseteq \mathbb{R}^d$ then $\mathcal{H}^s|_A$ is a Radon measure, where $\mathcal{H}^s|_A$ denotes the restriction of $\mathcal{H}^s$ to A. However, for Hausdorff measure we can state a stronger version of Vitali's covering theorem.

Theorem 2.16. *Let $H \subset \mathbb{R}^d$ be a $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(H) < \infty$ and $\mathcal{A}$ be a Vitali cover of H. Then there exists a disjoint sequence of sets (finite or countable) $A_1, A_2, \ldots \in \mathcal{A}$ such that either $\mathcal{H}^s\left(H \setminus \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = 0$ or $\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s = \infty$.*

For the proof see [16, Theorem 1.10].

The next two propositions are covering theorems stated in the from that we require later.

Proposition 2.17. *Let $H \subset \mathbb{R}^d$ be a $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(H) < \infty$ and $B \subset \mathbb{R}^d$ be a closed set with $0 < \operatorname{diam}(B) < \infty$ and $0 < \mathcal{H}^s(B) < \infty$. Let $\mathcal{A}$ be a Vitali cover of H such that every element of $\mathcal{A}$ is similar to B and every element of $\mathcal{A}$ is contained in H. Then there exists a disjoint sequence of sets (finite or countable) $A_1, A_2, \ldots \in \mathcal{A}$ such that $\mathcal{H}^s\left(H \setminus \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = 0$.*

Proof. Proposition 2.17 follows from Theorem 2.16 because $\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s = \infty$ is not possible since by the similarity it follows from Lemma 2.14 that $\frac{\operatorname{diam}(A_i)^s}{\operatorname{diam}(B)^s} = \frac{\mathcal{H}^s(A_i)}{\mathcal{H}^s(B)}$ and thus

$$\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s = \sum_{i=1}^{\infty} \mathcal{H}^s(A_i) \cdot \frac{\operatorname{diam}(B)^s}{\mathcal{H}^s(B)} \leq \mathcal{H}^s(H) \cdot \frac{\operatorname{diam}(B)^s}{\mathcal{H}^s(B)} < \infty.$$

□

Proposition 2.18. *Let $H \subset \mathbb{R}^d$ be a $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(H) < \infty$ and $B_1, \ldots, B_m \subset \mathbb{R}^d$ be closed sets with $0 < \operatorname{diam}(B_i) < \infty$ and $0 < \mathcal{H}^s(B_i) < \infty$ for all $i \in \{1, \ldots, m\}$. Let $\mathcal{A}$ be a Vitali cover of H such that every element of $\mathcal{A}$ is similar to B_i for some $i \in \{1, \ldots, m\}$ and every element of $\mathcal{A}$ is a subset of H. Then there exists a disjoint sequence of sets (finite or countable) $A_1, A_2, \ldots \in \mathcal{A}$ such that $\mathcal{H}^s\left(H \setminus \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = 0$.*

Proof. Assume that $A_1, A_2, \ldots \in \mathcal{A}$ is a disjoint sequence of sets. Let $M = \max_{1\le i\le m} \frac{\operatorname{diam}(B_i)^s}{\mathcal{H}^s(B_i)}$. If A_i is similar to B_j then

$$\operatorname{diam}(A_i)^s = \mathcal{H}^s(A_i)\frac{\operatorname{diam}(B_j)^s}{\mathcal{H}^s(B_j)} \le \mathcal{H}^s(A_i) \cdot M.$$

Hence

$$\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s = \sum_{i=1}^{\infty} \mathcal{H}^s(A_i) \cdot M \le \mathcal{H}^s(H) \cdot M < \infty.$$

Thus the proposition follows from Theorem 2.16. □

2.1.3 Other dimensions

Packing measure and dimension were introduced by Tricot [73] as a dual concept to Hausdorff measure and dimension. For $H \subseteq \mathbb{R}^d$, $s \in [0,\infty)$ and $\delta > 0$ let the *s-dimensional δ-approximate packing pre-measure of H* be

$$\mathcal{P}^s_\delta(H) = \sup\left\{\sum_{n=1}^{\infty} \operatorname{diam}(U_n)^s : \{U_n\}_{n=1}^{\infty} \text{ is a countable collection of pairwise disjoint}\right.$$
$$\left.\text{balls centred in } H \text{ with } \operatorname{diam}(U_n) \le \delta \text{ for all } n\right\}$$

and the *s-dimensional packing pre-measure of H* be

$$\mathcal{P}^s_0(H) = \inf_{\delta>0} \mathcal{P}^s_\delta(H) = \lim_{\delta\to 0} \mathcal{P}^s_\delta(H).$$

To ensure countable subadditivity, the *s-dimensional packing outer measure of H* is defined by

$$\mathcal{P}^s\,(H) = \inf\left\{\sum_{n=1}^{\infty} \mathcal{P}^s_0(H_i) : H \subseteq \bigcup_{n=1}^{\infty} H_n\right\}.$$

It follows from the definitions that for every $\delta > 0$

$$\mathcal{P}^s(H) \le \mathcal{P}^s_0(H) \le \mathcal{P}^s_\delta(H). \tag{2.3}$$

We say that an $M \subseteq \mathbb{R}^d$ is *$\mathcal{P}^s$-measurable* if

$$\mathcal{P}^s(H) = \mathcal{P}^s(H \cap M) + \mathcal{P}^s(H \setminus M)$$

for every $H \subseteq \mathbb{R}^d$ and we say that the *packing measure of M* is $\mathcal{P}^s(M)$. Let $\mathcal{M}(\mathcal{P}^s)$ denote the set of all $\mathcal{P}^s$-measurable sets of $\mathbb{R}^d$.

Theorem 2.19. *We have that $\mathcal{M}(\mathcal{P}^s)$ is a σ-algebra, $\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{M}(\mathcal{P}^s) \subseteq \mathcal{P}(\mathbb{R}^d)$, $\mathcal{P}^s$ is an outer measure on $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}^s$ is a measure on $\mathcal{M}(\mathcal{P}^s)$.*

For the proof see [52, Theorem 4.2] and [68, Theorem 3].

Theorem 2.20. *We have that* $\mathcal{H}^s(H) \leq \mathcal{P}^s(H)$ *for every* $H \subseteq \mathbb{R}^d$ *and* $s \geq 0$.

For details see [52, Theorem 5.12]

Theorem 2.21. *There exists* $0 < c_d < \infty$ *such that* $\mathcal{P}^d(H) = c_d \cdot \mathcal{L}^d(H)$ *for every* $H \subseteq \mathbb{R}^d$.

For the proof see [52, Theorem 3.4].

Theorem 2.22. *For every* $A \subseteq \mathbb{R}^d$ *there exists a unique* $s \in [0, d]$ *such that*
i) $t < s \Rightarrow \mathcal{P}^t(A) = \infty$
ii) $s < t \Rightarrow \mathcal{P}^t(A) = 0$.

For the proof see [52, Theorem 5.11].

Notation 2.23. We call this unique s the *packing dimension of* A and we denote it by $\dim_P A$.

Two further, widely studied definitions of dimension are the box dimensions. They are in some sense less sophisticated then the Hausdorff and packing dimensions. They cannot be defined via measures and they do not have such nice properties as Hausdorff and packing dimensions but they are still useful notions of Fractal Geometry. For a bounded set $H \subseteq \mathbb{R}^d$ let $N_\delta(H)$ be the minimum number of sets of diameter at most δ needed to cover H. The *upper box dimension of* H is

$$\overline{\dim}_B H = \limsup_{\delta \to 0} \frac{\log N_\delta(H)}{-\log \delta}$$

and the *lower box dimension of* H is

$$\underline{\dim}_B H = \liminf_{\delta \to 0} \frac{\log N_\delta(H)}{-\log \delta}.$$

When $\underline{\dim}_B H = \overline{\dim}_B H$ we say that the *box dimension of* H exists which we denote by $\dim_B H = \underline{\dim}_B H = \overline{\dim}_B H$.

As we can see from the following theorem, packing dimension is the countably stabilised version of the upper box dimension.

Theorem 2.24. *For* $H \subseteq \mathbb{R}^d$ *we have that*

$$\dim_P H = \inf \left\{ \sup_n \overline{\dim}_B H_n : H \subseteq \bigcup_{n=1}^{\infty} H_n \right\}.$$

For the proof see [52, Theorem 5.11].

Theorem 2.25. *For* $A \subseteq \mathbb{R}^d$ *we have that*

$$\dim_H A \leq \overline{\dim}_B A \leq \dim_P A$$

and

$$\dim_H A \leq \underline{\dim}_B A \leq \dim_P A.$$

For details see [14, page 46 (3.17)] and [14, page 52 (3.29)].

Proposition 2.26. *When substituting* dim *with any of* $\dim_H$, $\underline{\dim}_B$, $\overline{\dim}_B$ *or* $\dim_P$ *the following hold:*

i) Let $A \subseteq B \subseteq \mathbb{R}^d$ *then* $\dim(A) \leq \dim(B)$,

ii) Let $H \subseteq \mathbb{R}^d$ *and* $f : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ $(d_2 \in \mathbb{N})$ *be a Lipschitz map then* $\dim(f(H)) \leq \dim(H)$,

iii) Let $H \subseteq \mathbb{R}^d$ *and* $f : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ $(d_2 \in \mathbb{N})$ *be a bi-Lipschitz map then* $\dim_H(f(H)) = \dim_H(H)$

For details see [14, page 48].

Proposition 2.27. *Let* $A_n \subseteq \mathbb{R}^d$ $(n \in \mathbb{N})$ *be a sequence of sets. When substituting* dim *with either* $\dim_H$ *or* $\dim_P$ *then*

$$\dim \bigcup_{n \in \mathbb{N}} A_n = \sup_{n \in \mathbb{N}} \dim A_n.$$

Proposition 2.27 follows easily from the definitions of Hausdorff and Packing dimensions.

Proposition 2.28. *Let* $A_1, \ldots, A_N \subseteq \mathbb{R}^d$ *for some* $N \in \mathbb{N}$, *then*

$$\overline{\dim}_B \bigcup_{n=1}^{N} A_n = \sup_{n=1,\ldots,N} \overline{\dim}_B A_n.$$

For details see [14, page 48 (iii)].

For product sets it is not the case that $\mathcal{H}^{s+t} = \mathcal{H}^s \times \mathcal{H}^t$. However, some Fubini-like inequialities still hold.

Theorem 2.29. *Let* $A, B \subseteq \mathbb{R}^d$ *be non-empty Borel sets. Then*

i) $\mathcal{H}^{s+t}(A \times B) > 0$ *if* $\mathcal{H}^s(A) > 0$ *and* $\mathcal{H}^t(B) > 0$,

ii) $\dim_H A + \dim_H B \leq \dim_H A \times B \leq \dim_H A + \dim_P B$,

iii) $\dim_P A \times B \leq \dim_P A + \dim_P B$

For details see [52, Theorem 8.10].

2.2 Self-similar structures

In this section we introduce the classes of object that we discuss in this study. They all exhibit some sort of deterministic repetition on every scale. The three main classes are self-similar sets, attractors of graph directed iterated functions systems and subshifts of finite type.

A Lipschitz function $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ with Lipschitz constant strictly less than 1 is called a *contraction*.

Theorem 2.30. *Let $S_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be contractions for $1 \leq i \leq m$. Then there exists a unique non-empty compact K such that*

$$K = \bigcup_{i=1}^{m} S_i(K).$$

For the complete proof see [41, 3.2], however, we sketch the main idea of the proof. We call a finite family $\{S_i : 1 \leq i \leq m\}$ of contractions of $\mathbb{R}^d$ an *iterated function system* (IFS) in $\mathbb{R}^d$. The unique K that satisfies

$$K = \bigcup_{i=1}^{m} S_i(K) \tag{2.4}$$

is called the *attractor* of the IFS $\{S_i : 1 \leq i \leq m\}$. For $r > 0$ and $x \in \mathbb{R}^d$ let $B(x,r) = \left\{y \in \mathbb{R}^d : \|x - y\| < r\right\}$. For $r > 0$ and $H \subseteq \mathbb{R}^d$ we denote the *r-neigbourhood* of H by $B(H,r)$, i.e. $B(H,r) = \left\{x \in \mathbb{R}^d : \exists y \in H, \|x-y\| < r\right\}$. For two sets $A, B \subseteq \mathbb{R}^d$ we define the *Hausdorff distance between A and B* by

$$d_H(A,B) = \inf\left\{\varepsilon > 0 : B \subseteq B(A,\varepsilon) \,\text{and}\, A \subseteq B(B,\varepsilon)\right\}.$$

Let $\mathcal{K}^d$ denote the set of all non-empty compact subsets of $\mathbb{R}^d$. Then $(\mathcal{K}^d, d_H)$ is a complete metric space [12, Theorem 2.5.3]. To prove Theorem 2.30 it is sufficient to show that the set operation $H \longmapsto \bigcup_{i=1}^{m} S_i(H)$ has a unique fixed point K in $\mathcal{K}^d$. Thus the proof proceeds by showing that $H \longmapsto \bigcup_{i=1}^{m} S_i(H)$ is a contraction of $(\mathcal{K}^d, d_H)$ and by applying Banach's fixed point theorem there exists a unique fixed point K . Hence starting with any non-empty compact set K_0, defining a sequence recursively as $K_{n+1} = \bigcup_{i=1}^{m} S_i(K_n)$, the sequence converges to K in $(\mathcal{K}^d, d_H)$.

2.2.1 Self-similar sets

A self-similar set is such a set that is made up of its own scaled down copies where the scaling functions are all contractive similarities. It was first considered by Moran [57, Theorem II] for sets where the scaled down copies are disjoint.

A *self-similar iterated function system* (SS-IFS) in $\mathbb{R}^d$ is a finite collection of maps $\{S_i\}_{i=1}^{m}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that all the S_i are contracting similarities. The attractor of an SS-IFS is called a *self-similar set*.

We say that the SS-IFS $\{S_i\}_{i=1}^{m}$ satisfies the *strong separation condition* (SSC) if the $\{S_i(K)\}_{i=1}^{m}$ are disjoint. We say that the SS-IFS $\{S_i\}_{i=1}^{m}$ satisfies the *open set condition* (OSC) if there exists a nonempty open set $U \subseteq \mathbb{R}^d$ such that

$$\bigcup_{i=1}^{m} S_i(U) \subseteq U$$

and the union is disjoint. It is easy to see that SSC implies OSC.

Let $\{S_i\}_{i=1}^{m}$ be an SS-IFS. Then every S_i can be uniquely decomposed as

$$S_i(x) = r_i T_i(x) + v_i \tag{2.5}$$

for all $x \in \mathbb{R}^d$, where $0 < r_i < 1$, T_i is an orthogonal transformation and $v_i \in \mathbb{R}^d$ is a translation vector, for all indices i. The unique solution s of the equation

$$\sum_{i=1}^{m} r_i^s = 1 \tag{2.6}$$

is called the *similarity dimension* of the SS-IFS. It is well-known that if the SS-IFS satisfies the OSC then $0 < \mathcal{H}^s(K) < \infty$. Let $\mathcal{T}$ denote the group generated by the orthogonal transformations $\{T_i\}_{i=1}^m$. We call $\mathcal{T}$ the *transformation group* of the SS-IFS.

We denote the set $\{1, 2, \ldots, m\}$ by $\mathcal{I}$. Let $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ i.e. a k-tuple of indices. Then we write $S_{\mathbf{i}} = S_{i_1} \circ \ldots \circ S_{i_k}$ and $K_{\mathbf{i}} = S_{\mathbf{i}}(K)$. Since the similarities are decomposed as in (2.5) we write $r_{\mathbf{i}} = r_{i_1} \cdot \ldots \cdot r_{i_k}$ and $T_{\mathbf{i}} = T_{i_1} \circ \ldots \circ T_{i_k}$. For $\mathbf{i} = (i_1, \ldots, i_{k_1}), \mathbf{j} = (j_1, \ldots, j_{k_2}) \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ let $\mathbf{i} * \mathbf{j} = (i_1, \ldots, i_{k_1}, j_1, \ldots, j_{k_2})$.

Proposition 2.31. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and s be the similarity dimension of $\{S_i\}_{i=1}^m$. Then for every $\mathcal{H}^s$-measurable subset $A \subseteq K$ we have that $\mathcal{H}^s(A) = \mathcal{H}^s_\infty(A) \leq \operatorname{diam}(A)^s$. In particular, $\mathcal{H}^s(K) \leq \operatorname{diam}(K)^s < \infty$.*

For details see [2, Proposition 3].

We say that $\mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ are *comparable* if there exists $\mathbf{f} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that either $\mathbf{i} * \mathbf{f} = \mathbf{j}$ or $\mathbf{j} * \mathbf{f} = \mathbf{i}$. We say that $\mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ are *incomparable* if they are not comparable.

Proposition 2.32. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K, let s be the similarity dimension of $\{S_i\}_{i=1}^m$ and let $\mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ be incomparable. Then $\mathcal{H}^s(K_{\mathbf{i}} \cap K_{\mathbf{j}}) = 0$.*

For the details see [2, Proposition 3].

Proposition 2.33. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K. Then $\dim_H(K) = \underline{\dim}_B(K) = \overline{\dim}_B(K) = \dim_P(K)$ and $\mathcal{H}^t(K) < \infty$ where $t = \dim_H(K)$.*

Proposition 2.33 can be deduced by implicit methods [15, Thm 3.2].

Theorem 2.34. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and let s be the similarity dimension of $\{S_i\}_{i=1}^m$. Then the following are equivalent:*

(i) $\{S_i\}_{i=1}^m$ satisfies the OSC,

(ii) $\dim_H(K) = s$ and $0 < \mathcal{H}^s(K) < \infty$,

(iii) $0 < \mathcal{H}^s(K)$.

Remark 2.35. For an SS-IFS $\{S_i\}_{i=1}^m$ and $0 \leq p_1, \ldots, p_m \leq 1$ with $\sum_{i=1}^m p_i = 1$ there exists a unique Borel probability measure μ on $\mathbb{R}^d$ such that for every Borel set A we have that

$$\mu(A) = \sum_{i=1}^{m} p_i \cdot \mu(S_i^{-1}(A)),$$

(see for example [15, Theorem 2.8]). We call such a measure a *self-similar measure*. An interesting consequence of Theorem 2.34 that if the OSC is satisfied then $\mu = \frac{\mathcal{H}^s|_K(.)}{\mathcal{H}^s(K)}$ is a self-similar measure with $p_i = r_i^s$ for $i = 1, \ldots, m$, where $\mathcal{H}^s|_K(.)$ denotes the restriction of $\mathcal{H}^s$ onto K.

Corollary 2.36. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^d$ with attractor K and let d be the similarity dimension of $\{S_i\}_{i=1}^m$. Then the following are equivalent:*

(i) K has a nonempty interior,

(ii) $0 < \mathcal{H}^d(K)$.

For details of the proof of Theorem 2.34 and Corollary 2.36 see [70].

When the similarity dimension is greater than d then it is possible to have $0 < \mathcal{H}^d(K)$ while the interior of K is empty. Csörnyei, Jordan, Pollicott, Preiss and Solomyak [10] provided a family of such sets in the plane. Whether such a self-similar set exists in $\mathbb{R}$ is still an open problem.

If $\mathcal{F}$ is a set of linear transformations of $\mathbb{R}^d$ then by the *closure of $\mathcal{F}$* we mean the closure in the set of all linear transformations of $\mathbb{R}^d$ equipped with the Euclidean operator norm metric. We denote the closure of $\mathcal{F}$ by $\overline{\mathcal{F}}$.

Theorem 2.37. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K. Then the following are equivalent:*

(i) $\{S_i\}_{i=1}^m$ satisfies the OSC,

(ii) the identity map is not in the closure of $\left\{S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k, \mathbf{i} \neq \mathbf{j}\right\}$.

Theorem 2.37 follows from [2] and Theorem 2.34.

We say that the SS-IFS $\{S_i\}_{i=1}^m$ satisfies the *weak separation property* if the identity map is an isolated point of the set

$$\left\{S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k\right\}. \tag{2.7}$$

Iterating $K = \bigcup_{i=1}^m S_i(K)$ gives

$$K = \bigcup_{\mathbf{i} \in \mathcal{I}^k} S_{\mathbf{i}}(K) = \bigcup_{\mathbf{i} \in \mathcal{I}^k} K_{\mathbf{i}} \tag{2.8}$$

for every positive integer k. If s is the similarity dimension of $\{S_i\}_{i=1}^m$ then

$$\sum_{\mathbf{i} \in \mathcal{I}^k} r_{\mathbf{i}}^s = \left(\sum_{i=1}^m r_i^s\right)^k = 1$$

by (2.6). Thus K is the attractor of the SS-IFS $\left\{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ and the similarity dimension of $\left\{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ is s.

We say that there is an *exact overlap* in the SS-IFS $\{S_i\}_{i=1}^m$ if there exists $\mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$, $\mathbf{i} \neq \mathbf{j}$ such that $S_{\mathbf{i}} = S_{\mathbf{j}}$. This means that the semigroup action of the SS-IFS is not free.

Lemma 2.38. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and let s be the similarity dimension of $\{S_i\}_{i=1}^m$. If there is an exact overlap in $\{S_i\}_{i=1}^m$ then $\dim_H(K) < s$*

Proof. Assume that $\mathbf{i} \in \mathcal{I}^{k_1}$ and $\mathbf{j} \in \mathcal{I}^{k_2}$ for some $k_1, k_2 \in \mathbb{N}$ such that $\mathbf{i} \neq \mathbf{j}$ but $S_{\mathbf{i}} = S_{\mathbf{j}}$. Let $k = k_1 + k_2$. Then $S_{\mathbf{i}*\mathbf{j}} = S_{\mathbf{j}*\mathbf{i}}$ and hence K is the attractor of the SS-IFS $\left\{S_{\mathbf{f}} : \mathbf{f} \in \mathcal{I}^k, \mathbf{f} \neq \mathbf{i} * \mathbf{j}\right\}$. However

$$\sum_{\mathbf{f} \in \mathcal{I}^k, \mathbf{f} \neq \mathbf{i}*\mathbf{j}} r_{\mathbf{f}}^s = 1 - r_{\mathbf{i}*\mathbf{j}}^s < 1$$

thus the similarity dimension of $\left\{S_{\mathbf{f}} : \mathbf{f} \in \mathcal{I}^k, \mathbf{f} \neq \mathbf{i} * \mathbf{j}\right\}$ is strictly less than s. Hence $\dim_H(K) < s$ by Proposition 2.31. □

Proposition 2.39. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K. Then the following are equivalent:*

i) $\{S_i\}_{i=1}^m$ satisfies the weak separation property and there is no exact overlap in $\{S_i\}_{i=1}^m$,

ii) $\{S_i\}_{i=1}^m$ satisfies the OSC

Proof. Assume that *i)* holds. Since the weak separation property is satisfied the identity map is an isolated point of $\left\{S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k\right\}$. Since there is no exact overlap in $\{S_i\}_{i=1}^m$ it follows that $\left\{S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k, \mathbf{i} \neq \mathbf{j}\right\}$ does not contain the identity map. Hence the closure of $\left\{S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k, \mathbf{i} \neq \mathbf{j}\right\}$ does not contain the identity map. Thus $\{S_i\}_{i=1}^m$ satisfies the OSC by Theorem 2.37.

Now assume that *ii)* holds, i.e. $\{S_i\}_{i=1}^m$ satisfies the OSC. Then it follows from Theorem 2.37 that $\{S_i\}_{i=1}^m$ satisfies the weak separation property. By Theorem 2.34 $\dim_H(K)$ is the similarity dimension of $\{S_i\}_{i=1}^m$ and hence by Lemma 2.38 there is no exact overlap in $\{S_i\}_{i=1}^m$. □

2.2.2 Graph directed attractors

Graph directed attractors are a generalisation of attractors of IFSs and were introduced by Mauldin and Williams [56]. A graph directed attractor is a finite family of sets such that every set in the family is made up of finitely many scaled down copies of the members of the family.

Let $G(\mathcal{V}, \mathcal{E})$ be a directed graph, where $\mathcal{V} = \{1, 2, \ldots, q\}$ is the set of vertices and $\mathcal{E}$ is the finite set of directed edges such that for each $i \in \mathcal{V}$ there exists $e \in \mathcal{E}$ starting from i. Let $\mathcal{E}_{i,j}$ denote the set of edges from vertex i to vertex j and $\mathcal{E}_{i,j}^k$ denote the set of sequences of k edges $(e_1, \ldots, e_k)$ which form a directed path from vertex i to vertex j. A *graph directed iterated function system* (GD-IFS) in $\mathbb{R}^d$ is a finite collection of maps $\{S_e : e \in \mathcal{E}\}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that all the S_e are contractions which we will hencefort assume to be contracting similarties. The *attractor* of the GD-IFS is the unique q-tuple of nonempty compact sets $(K_1, \ldots, K_q)$ such that

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(K_j), \tag{2.9}$$

see [12, Theorem 4.3.5]. The attractor of a GD-IFS is called a *graph directed attractor*.

Let $\{S_e : e \in \mathcal{E}\}$ be a GD-IFS. Then every S_e can be uniquely decomposed as

$$S_e(x) = r_e T_e(x) + v_e \tag{2.10}$$

for all $x \in \mathbb{R}^d$, where $0 < r_e < 1$, T_e is an orthogonal transformation and $v_e \in \mathbb{R}^d$ is a translation vector, for all edges e. Let $A^{(s)}$ be the $q \times q$ matrix with (i,j)th entry given by

$$A^{(s)}_{i,j} = \sum_{e \in \mathcal{E}_{i,j}} r_e^s. \tag{2.11}$$

For a matrix A let $\rho(A)$ denote the spectral radius of A, that is the largest absolute value of the eigenvalues of A. The unique solution s of the equation

$$\rho(A^{(s)}) = 1 \tag{2.12}$$

is called the *similarity dimension* of the GD-IFS.

Let $\mathbf{e} = (e_1, \ldots, e_k) \in \mathcal{E}^k_{i,j}$, then we write $S_{\mathbf{e}}$ for $S_{e_1} \circ \ldots \circ S_{e_k}$ and $K_{\mathbf{e}}$ for $S_{\mathbf{e}}(K_j) \subseteq K_i$. If the similarities are decomposed as in (2.10) then we write $r_{\mathbf{e}}$ for $r_{e_1} \cdot \ldots \cdot r_{e_k}$ and $T_{\mathbf{e}}$ for $T_{e_1} \circ \ldots \circ T_{e_k}$. If $\mathbf{e} = (e_1, \ldots, e_k) \in \mathcal{E}^k_{i,j}$ and $\mathbf{f} = (f_1, \ldots, f_n) \in \mathcal{E}^n_{j,l}$ for $i, j, l \in \mathcal{V}$ then we write $\mathbf{e} * \mathbf{f}$ for $(e_1, \ldots, e_k, f_1, \ldots, f_n) \in \mathcal{E}^{k+n}_{i,l}$.

The directed graph $G(\mathcal{V}, \mathcal{E})$ is called *strongly connected* if for every pair of vertices i and j there exist a directed path from i to j and a directed path from j to i. We say that the GD-IFS $\{S_e : e \in \mathcal{E}\}$ is *strongly connected* if $G(\mathcal{V}, \mathcal{E})$ is strongly connected.

The set $\mathcal{C}_i := \bigcup_{k=1}^{\infty} \mathcal{E}^k_{i,i}$ is the set of directed cycles of $G(\mathcal{V}, \mathcal{E})$ that start and end in vertex i. Equipped with the $*$ operation $\mathcal{C}_i$ becomes a semigroup. Let $\mathcal{T}_{i,G}$ denote the group generated by the transformations

$$\{T_{e_1} \circ \ldots \circ T_{e_k} : (e_1, \ldots, e_k) \in \mathcal{C}_i\}, \tag{2.13}$$

we call $\mathcal{T}_{i,G}$ the *i-th transformation group* of the GD-IFS. It is easy to see that if the GD-IFS is strongly connected then $\mathcal{T}_{i,G}$ is conjugate to $\mathcal{T}_{j,G}$ for all $i, j \in \mathcal{V}$ and hence

$$|\mathcal{T}_{i,G}| = |\mathcal{T}_{j,G}|,$$

where $|.|$ denotes the cardinality of a set.

We say that the GD-IFS $\{S_e : e \in \mathcal{E}\}$ satisfies the *strong separation condition* (SSC) if the sets

$$\{S_e(K_j) : j \in \mathcal{V}, e \in \mathcal{E}_{i,j}\}$$

are disjoint for each $i \in \mathcal{V}$. We say that the GD-IFS $\{S_e : e \in \mathcal{E}\}$ satisfies the *open set condition* (OSC) if there exists a q-tuple of nonempty open sets $(U_1, \ldots, U_q)$ such that

$$\bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(U_j) \subseteq U_i$$

and the union is disjoint for each $i \in \mathcal{V}$. It is easy to see that SSC implies OSC for GD-IFSs.

We say that K is a *graph directed self-similar set* if there exists a GD-IFS with attractor $(K_1, \dots, K_q)$ such that $K = K_i$ for some $i \in \mathcal{V}$. Every self-similar set is a graph directed self-similar set. To see this, let K be the attractor of the SS-IFS $\{S_i\}_{i=1}^m$. Then let $\mathcal{V} = \{1\}$ and let $\mathcal{E} = \mathcal{I}$ such that every element of $\mathcal{E}$ is a loop on vertex $1 \in \mathcal{V}$. Then the 1-tuple (K) is the attractor of the GD-IFS $\{S_e : e \in \mathcal{E}\}$ associated to the graph $G(\mathcal{V}, \mathcal{E})$. Boore and Falconer [8] proved that graph-directed self-similar sets are genuinely more general than just self-similar sets. They showed that there exists a strongly connected GD-IFS in $\mathbb{R}$, on two vertices, with attractor (K_1, K_2) such that neither of K_1 or K_2 is a self-similar set.

Proposition 2.40. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS with attractor $(K_1, \dots, K_q)$ and let s be the similarity dimension of $\{S_e : e \in \mathcal{E}\}$. Then $\mathcal{H}^s(K_i) < \infty$ for $i \in \mathcal{V}$.*

For the details of the proof of Proposition 2.40 see [12, p. 204].

Proposition 2.41. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS with attractor $(K_1, \dots, K_q)$, let s be the similarity dimension of $\{S_e : e \in \mathcal{E}\}$ and $e, f \in \bigcup_{j=1}^q \mathcal{E}_{i,j}$, $e \neq f$ for some $i \in \mathcal{V}$. Then $\mathcal{H}^s(K_e \cap K_{\boldsymbol{f}}) = 0$.*

For the proof see [75, Proposition 2].

Proposition 2.42. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS with attractor $(K_1, \dots, K_q)$. Then $\dim_H(K_j) = \dim_H(K_i) = \underline{\dim}_B(K_i) = \overline{\dim}_B(K_i) = \dim_P(K_i)$ for each $i, j \in \mathcal{V}$ and $\mathcal{H}^t(K_i) < \infty$ for each $i \in \mathcal{V}$ where $t = \dim_H(K_i)$.*

Proposition 2.42 can be deduced by implicit methods [15, Theorem 3.2].

Theorem 2.43. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS with attractor $(K_1, \dots, K_q)$ and let s be the similarity dimension of $\{S_e : e \in \mathcal{E}\}$. Then the following are equivalent:*
(i) $\{S_e : e \in \mathcal{E}\}$ satisfies the OSC,
(ii) $\dim_H(K_i) = s$ and $0 < \mathcal{H}^s(K_i) < \infty$ for each $i \in \mathcal{V}$,
(iii) $0 < \mathcal{H}^s(K_i)$ for some $i \in \mathcal{V}$.

For the details of the proof of Theorem 2.43 see [75].

Irreducible matrices Recall that for a GD-IFS $\{S_e : e \in \mathcal{E}\}$ we define $A^{(s)}$ as in (2.11) $A^{(s)}_{i,j} = \sum_{e \in \mathcal{E}_{i,j}} r_e^s$. Then for the kth power of $A^{(s)}$ it follows that

$$\left(A^{(s)}\right)^k_{i,j} = \sum_{\mathbf{e} \in \mathcal{E}^k_{i,j}} r_{\mathbf{e}}^s$$

for all $i, j \in \mathcal{V}$. Thus $G(\mathcal{V}, \mathcal{E})$ is strongly connected if and only if for all $i, j \in \mathcal{V}$ there exists a positive integer k such that $\left(A^{(s)}\right)^k_{i,j} > 0$.

A $q \times q$ real matrix $A = (A_{i,j})$ is called *non-negative* and we write $A \geq 0$ if $A_{i,j} \geq 0$ for all $1 \leq i, j \leq q$. If $A_{i,j} > 0$ holds for all indices i, j then A is called *positive* and we write $A > 0$. For matrices A and B we write $A \geq B$ if $A - B \geq 0$ and similarly we write $A > B$ if $A - B > 0$. Similar definitions and notations apply to vectors in $\mathbb{R}^q$.

A non-negative matrix $A \geq 0$ is called *irreducible* if for every $1 \leq i, j \leq q$ there exists a positive integer k such that $\left(A^k\right)_{i,j} > 0$. We note that k can be chosen such that $k \leq q$ (see for example [1, Lemma 1.1.2]). There are several equivalent definitions of irreducible matrices but this definition is convenient for us. For a GD-IFS $\{S_e : e \in \mathcal{E}\}$ we have that $A^{(s)} \geq 0$ and also that $A^{(s)}$ is irreducible if and only if $G(\mathcal{V}, \mathcal{E})$ is strongly connected. The following theorem is the well-known Perron-Frobenius theorem.

Theorem 2.44. *Let $A \geq 0$ be a $q \times q$ irreducible matrix. Then*

(i) there exist $y \in \mathbb{R}^q$, $y > 0$ and $\lambda_0 \in \mathbb{R}$, $\lambda_0 > 0$ such that $Ay = \lambda_0 y$,

(ii) the eigenvalue λ_0 is a simple root of the characteristic polynomial of A,

(iii) $\rho(A) = \lambda_0$,

(iv) the only non-negative, nonzero eigenvectors of A are the positive scalar multiples of y.

For details see [1, Thm 1.4.4].

Remark 2.45. Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS with attractor $(K_1, \ldots, K_q)$ and let s be the similarity dimension of $\{S_e : e \in \mathcal{E}\}$. Let $y_i = \mathcal{H}^s(K_i)$ and $y^\intercal = (y_1, \ldots, y_q)$. By Proposition 2.41 we have $\mathcal{H}^s(K_e \cap K_f) = 0$ for $e, f \in \bigcup_{j=1}^q \mathcal{E}_{i,j}$, $e \neq f$. Hence

$$y_i = \mathcal{H}^s(K_i) = \sum_{j=1}^{q} \sum_{e \in \mathcal{E}_{i,j}} \mathcal{H}^s(K_e) = \sum_{j=1}^{q} \sum_{e \in \mathcal{E}_{i,j}} r_e^s \cdot \mathcal{H}^s(K_j) = \sum_{j=1}^{q} A_{i,j}^{(s)} \cdot y_j$$

so

$$y = A^{(s)} y.$$

If $\{S_e : e \in \mathcal{E}\}$ satisfies the OSC then by Theorem 2.43 $y \in \mathbb{R}^q$, $y > 0$. In that case y satisfies Theorem 2.44 with $1 = \rho(A^{(s)}) = \lambda_0$.

Corollary 2.46. *Let $A \geq 0$ be a $q \times q$ irreducible matrix. If there exists a non-negative, non-zero vector $u \in \mathbb{R}^q$ such that $Au = u$ then $\rho(A) = 1$.*

Corollary 2.46 follows from Theorem 2.44.

Lemma 2.47. *Let $A \geq B \geq 0$ be $q \times q$ irreducible matrices such that $A \neq B$. Then $\rho(A) > \rho(B)$.*

Lemma 2.47 follows from [50, 5.7.5].

It follows from Lemma 2.47 that for a GD-IFS $\rho(A^{(s)})$ is strictly decreasing in s.

Lemma 2.48. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS with similarity dimension s and let $e_0 \in \mathcal{E}$ such that $\{S_e : e \in \mathcal{E} \setminus \{e_0\}\}$ is a strongly connected GD-IFS with similarity dimension s_0. Then $s_0 < s$.*

Lemma 2.48 follows from Lemma 2.47.

2.2.3 Subshifts of finite type

Let $\mathcal{J} = \{0, \ldots, M-1\}$ be a finite alphabet, let $\Sigma = \mathcal{J}^{\mathbb{N}}$. We will write $\mathbf{i} = (i_0, \ldots, i_{k-1}) \in \mathcal{J}^k$ and $\alpha = (\alpha_0, \alpha_1, \ldots) \in \Sigma$. We will also write $\alpha|_k = (\alpha_0, \ldots, \alpha_{k-1}) \in \mathcal{J}^k$ for the restriction of α to its first k coordinates. We equip Σ with the standard metric defined by

$$d(\alpha, \beta) = 2^{-n(\alpha,\beta)}$$

for $\alpha \neq \beta$, where $n(\alpha, \beta) = \max\{n \in \mathbb{N} : \alpha|_n = \beta|_n\}$. We write $\mathcal{J}^* = \bigcup_{k\in\mathbb{N}} \mathcal{J}^k$ for the set of all finite words. For $\mathbf{i} = (i_0, \ldots, i_{k-1}) \in \mathcal{J}^*$, we write

$$[\mathbf{i}] = \big\{\alpha \in \Sigma : \alpha|_k = \mathbf{i}\big\}$$

for the *cylinder* corresponding to $\mathbf{i}$ and we let $|\mathbf{i}| = k$ be the length of $\mathbf{i}$. Let $\sigma : \Sigma \to \Sigma$ be the *one-sided left shift* i.e. $\sigma(\alpha) = (\alpha_1, \alpha_2, \ldots)$ for $\alpha = (\alpha_0, \alpha_1, \ldots) \in \Sigma$. Even though the shift is only defined on Σ, it will be convenient also to define it for $\mathbf{i} = (i_0, \ldots, i_{k-1}) \in \mathcal{J}^*$ by

$$\sigma(\mathbf{i}) = \sigma\big((i_0, \ldots, i_{k-1})\big) = (i_1, \ldots, i_{k-1}).$$

Any closed σ-invariant set $\Lambda \subseteq \Sigma$ is called a *subshift*. Among the most important subshifts are subshifts of finite type which we define as follows. Let A be an $M \times M$ transition matrix indexed by $\mathcal{J} \times \mathcal{J}$ with entries in $\{0, 1\}$. We define the *subshift of finite type* corresponding to A as

$$\Sigma_A = \Big\{\alpha = (\alpha_0, \alpha_1, \ldots) \in \Sigma : A_{\alpha_i, \alpha_{i+1}} = 1 \text{ for all } i = 0, 1, \ldots\Big\}.$$

If every entry of A is 1 then we call $\Sigma_A = \Sigma$ the *full shift*. We say Σ_A is irreducible (or transitive) if the matrix A is irreducible. We say Σ_A is aperiodic (or mixing) if the matrix A is aperiodic, which means that there exists $n \in \mathbb{N}$ such that $(A^n)_{i,j} > 0$ for all pairs $i, j \in \mathcal{J}$ simultaneously.

To each $i \in \mathcal{J}$ associate a contracting similarity map $S_i(x) = r_i \cdot T_i(x) + v_i$ on $\mathbb{R}^d$ where $r_i \in (0, 1)$, T_i is an orthogonal transformation and $v_i \in \mathbb{R}^d$. For $\mathbf{i} = (i_0, \ldots, i_{k-1}) \in \mathcal{J}^*$ we write

$$S_{\mathbf{i}} = S_{i_0} \circ \cdots \circ S_{i_{k-1}},$$

$$T_{\mathbf{i}} = T_{i_0} \circ \cdots \circ T_{i_{k-1}}$$

and

$$r_{\mathbf{i}} = r_{i_0} \cdot \cdots \cdot r_{i_{k-1}}.$$

Then

$$\Pi(\alpha) = \lim_{k\to\infty} S_{\alpha|_k}(0)$$

exists for every $\alpha \in \Sigma$. For a given subshift of finite type Σ_A, we are interested in the set $F_A := \Pi(\Sigma_A) \subseteq \mathbb{R}^d$. We call this set F_A the *attractor* of the subshift of finite type Σ_A. The set $F := \Pi(\Sigma)$ corresponding to the full shift is a self-similar set since it satisfies

$$F = \bigcup_{i\in\mathcal{J}} S_i(F).$$

For $j \in \mathcal{J}$ let $F_A^j = \Pi(\Sigma_A^j)$ where $\Sigma_A^j = \{(\alpha_0, \alpha_1, \dots) \in \Sigma_A : \alpha_0 = j\}$. We will also be interested in subsets of F_A corresponding to cylinders of higher generations. For $\mathbf{i} \in \mathcal{I}^*$, let

$$F_A^{\mathbf{i}} = \Pi(\Sigma_A \cap [\mathbf{i}]),$$

which may be empty.

Equivalence of graph directed attractors and subshifts of finite type Here we show that graph directed attractors and subshifts of finite type are the same in some sense. Similar results were shown by Marcus and Lind [47, Proposition 2.2.6 and Proposition 2.3.9] and were stated in this form by Farkas and Fraser [26].

Proposition 2.49. *Let Σ_A be a subshift of finite type for the alphabet $\mathcal{J}$, with attractor F_A. Assume that A has at least one non-zero entry in every row. Then there exists a GD-IFS on a directed graph $G(\mathcal{J}, \mathcal{E})$, with attractor $(F_A^i)_{i \in \mathcal{J}}$. We have that A is irreducible if and only if $G(\mathcal{J}, \mathcal{E})$ is strongly connected.*

Proof. We draw a directed edge $e = e_{i,j}$ from i to j of $\mathcal{J}$ if $A_{i,j} = 1$, let $S_e = S_i$ and let $\mathcal{E} = \{e_{i,j} : i, j \in \mathcal{J}, A_{i,j} = 1\}$. It follows that A is irreducible if and only if the graph $G(\mathcal{J}, \mathcal{E})$ is strongly connected. We have that

$$F_A^i = \bigcup_{j \in \mathcal{J}, A_{i,j} = 1} S_i(F_A^j) = \bigcup_{j \in \mathcal{J}} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(F_A^j),$$

thus $(F_A^i)_{i \in \mathcal{J}}$ is the attractor of the GD-IFS $\{S_e : e \in \mathcal{E}\}$. $\square$

If the row in A corresponding to the letter $i \in \mathcal{J}$ contains only zero entries, i.e. every letter is forbidden after the letter i, then $F_A^i = \emptyset$. Thus the role of the assumption, that A has at least one non-zero entry in every row, is to rule out the empty first level sets. If we eliminate the letters from $\mathcal{J}$ which correspond to rows with only zero entries then we are left with a system that satisfies the assumption of Proposition 2.49 and we only excluded the empty set entries from $(F_A^i)_{i \in \mathcal{J}}$.

Proposition 2.50. *Let $(K_1, \dots, K_q)$ be the attractor of a GD-IFS with directed graph $G(\mathcal{V}, \mathcal{E})$. Then there exists a subshift of finite type Σ_A associated to the alphabet $\mathcal{J} = \mathcal{E}$ such that $F_A^e = S_e(K_j)$ for every $i, j \in \mathcal{V}$ and $e \in \mathcal{E}_{i,j}$. In particular, $K_i = \bigcup_{\{f \in \mathcal{E} : A_{e,f} = 1\}} F_A^f = S_e^{-1}(F_A^e)$ for every $i \in \mathcal{V}$ whenever $e \in \bigcup_{j=1}^q \mathcal{E}_{j,i}$. We have that $G(\mathcal{V}, \mathcal{E})$ is strongly connected if and only if the constructed subshift of finite type is irreducible.*

Proof. Let the alphabet be indexed by the edge set $\mathcal{E}$. Now, for two edges $e, f \in \mathcal{E}$, let $A_{e,f} = 1$ if and only if f begins from the vertex where e ends, i.e. it is possible to walk along e and then along f. It follows that $G(\mathcal{V}, \mathcal{E})$ is strongly connected if and only if the matrix A is irreducible. It is now straightforward to see that for all $i, j \in \mathcal{V}$ and $e \in \mathcal{E}_{i,j}$

$$F_A^e = S_e(K_j)$$

and so for all $i \in \mathcal{V}$ it follows that

$$K_i = \bigcup_{j=1}^{q} \bigcup_{f \in \mathcal{E}_{i,j}} F_A^f$$

and, moreover, for any edge e which finishes at i

$$\bigcup_{j=1}^{q} \mathcal{E}_{i,j} = \{f \in \mathcal{E} : A_{e,f} = 1\}.$$

Thus

$$K_i = \bigcup_{\{f \in \mathcal{E} : A_{e,f}=1\}} K_A^f$$

for $i \in \mathcal{V}$ and $e \in \bigcup_{j=1}^{q} \mathcal{E}_{j,i}$ as required. $\square$

Properties Relying on the analogy of Proposition 2.49 we proceed as follows. The set of directed cycles in the graph $G(\mathcal{J}, \mathcal{E})$ constructed in Proposition 2.49 is

$$\mathcal{C}_j = \bigcup_{k=1}^{\infty} \mathcal{E}_{j,j}^k = \bigcup_{k=1}^{\infty} \Big\{ (\alpha_0, \ldots, \alpha_{k-1}) \in \mathcal{J}^k : \alpha_0 = j, A_{\alpha_{k-1}, j} = 1$$
$$\text{and}\, A_{\alpha_i, \alpha_{i+1}} = 1 \,\text{for all}\, i = 0, 1, \ldots, k-2 \Big\}.$$

Hence we define the *j-th transformation group $\mathcal{T}_{j,A}^G$ of Σ_A* to be the group generated by the semigroup

$$\left\{T_\alpha : k \in \mathbb{N}, \alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in \mathcal{J}^k, \alpha_0 = j \,\text{and}\, A_{\alpha_i, \alpha_{i+1}} = 1 \,\text{for all}\, i = 0, 1, \ldots, k-2\right\}.$$

Let $A^{(s)}$ be the $\mathcal{J} \times \mathcal{J}$ matrix with (i, j)th entry given by

$$A_{i,j}^{(s)} = \begin{cases} r_i^s & \text{if } A_{i,j} = 1 \\ 0 & \text{if } A_{i,j} = 0 \end{cases}, \tag{2.14}$$

say that the unique s, such that

$$\rho(A^{(s)}) = 1,$$

is the *similarity dimension* of the subshift of finite type Σ_A.

We say that Σ_A satisfies the *strong separation condition* (SSC) if

$$\left\{F_A^j : j \in \mathcal{J}, A_{i,j} = 1\right\}$$

are disjoint for every $i \in \mathcal{J}$. We say that Σ_A satisfies the *open set condition* (OSC) if there exists an M-tuple $(U_0, \ldots, U_{M-1})$ of open sets of $\mathbb{R}^d$ such that

$$\bigcup_{j \in \mathcal{J}, A_{i,j}=1} S_i(U_j) \subseteq U_i$$

and the union is disjoint for every $i \in \mathcal{J}$.

Due to the construction in Proposition 2.49 we can reformulate results about GD-IFSs for subshifts of finite type. The following is the subshift of finite type analogue of Proposition 2.40.

Proposition 2.51. *Let Σ_A be an irreducible subshift of finite type, with attractor F_A and let s be the similarity dimension of Σ_A. Then $\mathcal{H}^s(F_A^i) < \infty$ for $i \in \mathcal{J}$ and thus $\mathcal{H}^s(F_A) < \infty$.*

We can also state the subshift of finite type analogue of Proposition 2.41.

Proposition 2.52. *Let Σ_A be an irreducible subshift of finite type with attractor F_A, let s be the similarity dimension of Σ_A and let $i, j, l \in \mathcal{J}$, $j \neq l$ be such that $A_{i,j} = 1$ and $A_{i,l} = 1$. Then $\mathcal{H}^s(F_A^j \cap F_A^l) = 0$.*

We can reformulate the subshift of finite type analogue of Proposition 2.42.

Proposition 2.53. *Let Σ_A be an irreducible subshift of finite type with attractor F_A. Then $\dim_H(F_A^j) = \dim_H(F_A^i) = \underline{\dim}_B(F_A^i) = \overline{\dim}_B(F_A^i) = \dim_P(F_A^i)$ for each $i, j \in \mathcal{V}$ and $\mathcal{H}^t(F_A^i) < \infty$ for each $i \in \mathcal{V}$ where $t = \dim_H(F_A^i)$. In particular, $\mathcal{H}^t(F_A) < \infty$ and $\dim_H(F_A) = t$.*

The next theorem is the subshift of finite type analogue of Theorem 2.43.

Theorem 2.54. *Let Σ_A be an irreducible subshift of finite type with attractor F_A and let s be the similarity dimension of Σ_A. Then the following are equivalent:*
(i) Σ_A satisfies the OSC,
(ii) $\dim_H(F_A^i) = s$ and $0 < \mathcal{H}^s(F_A^i) < \infty$ for each $i \in \mathcal{V}$,
(iii) $0 < \mathcal{H}^s(K_i)$ for some $i \in \mathcal{V}$.

Extension to k-block subshifts of finite type We only consider 2-block subshifts of finite type in this paper, i.e. where the forbidden words are of length 2, but note that our results can be extended to the more general k-block case, where the forbidden words are of length k. This is a natural simplification to make, as one can always reformulate a k-block subshift of finite type as a 2-block analogue over a larger alphabet. Moreover, this can be done so that for irreducible k-block systems the associated 2-block system remains irreducible. The reformulation is straightforward and standard. The new alphabet is the set of words of length $(k-1)$ such that there is an allowable word of length k beginning with that word of length $(k-1)$. Then, the 2-word (over the new alphabet) consisting of $(i_0, i_1, \ldots, i_{k-2})$ followed by $(i_1, i_2, \ldots, i_{k-1})$ is allowed if and only if $(i_0, i_1, \ldots, i_{k-1})$ was allowed in the original k-block system. For $(i_0, i_1, \ldots, i_{k-2})$ the associated similarity is S_{i_0}. Then the attractor of the 2-block system and the attractor of the k-block system are the same. A similar argument can be found in [47, Theorem 2.3.2].

2.3 Orthogonal transformations

It turns out that many geometric and measure theoretic properties of self-similar structures depend on their transformation groups. Hence we summarize useful facts about orthogonal transformations here.

A linear map $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is called an *orthogonal transformation* if it preserves the Euclidean norm, i.e. $\|T(y)\| = \|y\|$ for all $y \in \mathbb{R}^d$, hence $\|T\| = 1$. If L is linear and T is orthogonal then it follows that $\|L\| = \|L \circ T\|$. Similarly if $T : \mathbb{R}^{d_2} \longrightarrow \mathbb{R}^{d_2}$ is an orthogonal transformation and L is linear as above then $\|L\| = \|T \circ L\|$. We denote the set of all orthogonal transformations of $\mathbb{R}^d$ by $\mathbb{O}_d$. With the Euclidean operator norm metric $\mathbb{O}_d$ is a compact topological group. For $\mathcal{T} \subseteq \mathbb{O}_d$ we denote by $\overline{\mathcal{T}}$ the closure of $\mathcal{T}$ in this topology. Let $\mathbb{SO}_d$ denote the set of all special orthogonal transformations, i.e. the ones with determinant 1.

Lemma 2.55. *If $T \in \mathbb{O}_d$ then for every $\delta > 0$ there exists a positive integer k such that $\left\|T^k - Id_{\mathbb{R}^d}\right\| < \delta$.*

Proof. Lemma 2.55 follows from the compactness of the set of all orthogonal transformations. By compactness there exists a Cauchy subsequance $\left(T^{k_i}\right)_{i=1}^{\infty}$ of the sequance $(T_k)_{k=1}^{\infty}$. Thus for every $\delta > 0$ there exists a positive integer i such that $\left\|T^{k_{(i+1)}} - T^{k_i}\right\| < \delta$ and so for $k = k_{(i+1)} - k_i$ we have that $\left\|T^k - Id_{\mathbb{R}^d}\right\| = \left\|T^{k_{(i+1)}} - T^{k_i}\right\| < \delta$. $\square$

Lemma 2.56. *If $T_1, \ldots, T_m \in \mathbb{O}_d$ then the semigroup generated by $T_1, \ldots, T_m$ is dense in the group generated by $T_1, \ldots, T_m$.*

Lemma 2.56 follows from Lemma 2.55.

The next theorem is Kronecker's simultaneous approximation theorem.

Theorem 2.57. *If $1, \beta_1, \ldots, \beta_m \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$ and $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$, $N \in \mathbb{N}$, $\varepsilon_0 > 0$ then there exist $p_0 \in \mathbb{N}$, $k_1, \ldots, k_m \in \mathbb{Z}$ such that*

$$|p_0 \cdot \beta_i - \gamma_i - k_i| < \varepsilon_0$$

for $i = 1, \ldots m$ and $p_0 > N$.

For the details of the proof of Theorem 2.57 see [37, Theorem 442].

Corollary 2.58. *If $1, \beta_1, \ldots, \beta_m \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$, $\widehat{q} \in \mathbb{N}$, $\varepsilon > 0$ then there exist $p \in \mathbb{N}$, $d_1, \ldots, d_m \in \mathbb{Z}$ such that*

$$|p \cdot \beta_i - \alpha_i - d_i| < \varepsilon$$

for $i = 1, \ldots m$ where $p, d_1, \ldots, d_m$ are multiples of $\widehat{q}$.

Proof. We can apply Theorem 2.57 for $\gamma_i = \alpha_i/\widehat{q}$, $\varepsilon_0 = \varepsilon/\widehat{q}$ and $N = 1$ to get $p_0, k_1, \ldots, k_m$. Then $p = p_0 \cdot \widehat{q}$, $d_i = k_i \cdot \widehat{q}$ satisfy the statement. $\square$

The following proposition relies on Kronecker's simultaneous approximation theorem. We did not find this in the literature, so we provide the proof here.

Proposition 2.59. *If $T \in \mathbb{O}_d$ then for all $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$, $k \geq N$, such that the group generated by T^k is dense in the group generated by T.*

Proof. By [7, Theorem 10.12] we can find an orthonormal basis in $\mathbb{R}^d$ with respect to which the matrix form of T is block diagonal such that the blocks are either $\begin{bmatrix}1\end{bmatrix}$ or $\begin{bmatrix}-1\end{bmatrix}$ or $B(\alpha_i) = \begin{bmatrix}\cos(\alpha_i) & -\sin(\alpha_i)\\ \sin(\alpha_i) & \cos(\alpha_i)\end{bmatrix}$ for some $\alpha_1, \dots, \alpha_n \in [0, 2\pi)$. Let $\mathcal{J} = \{i \in \{1, \dots, n\} : B(\alpha_i) \text{ has finite order}\}$. If $i \in \mathcal{J}$ then let k_i be the order of $B(\alpha_i)$. Let $k_0 = 2^N \cdot \prod_{i \in \mathcal{J}} k_i > N$. Then for any $l \in \mathbb{N}$ it follows that $B(\alpha_i)^{l \cdot k_0 + 1} = B(\alpha_i)$ for all $i \in \mathcal{J}$, $\begin{bmatrix}1\end{bmatrix}^{l \cdot k_0 + 1} = \begin{bmatrix}1\end{bmatrix}$ and $\begin{bmatrix}-1\end{bmatrix}^{l \cdot k_0 + 1} = \begin{bmatrix}-1\end{bmatrix}$.

Let $\mathcal{A} = \left\{\frac{\alpha_i}{2\pi} : i \in \{1, \dots, n\} \setminus \mathcal{J}\right\}$. If $\frac{\alpha_i}{2\pi} \in \mathcal{A}$ then $\frac{\alpha_i}{2\pi}$ is irrational. Let $1, \frac{\beta_1}{2\pi}, \dots, \frac{\beta_m}{2\pi} \in \mathcal{A} \bigcup \{1\}$ be a maximal linearly independent system over $\mathbb{Q}$. Then we can write every α_i for $i \in \{1, \dots, n\} \setminus \mathcal{J}$ in the form $\alpha_i = \left(\frac{p_{i,0}}{q_{i,0}} \cdot 2\pi + \sum_{j=1}^m \frac{p_{i,j}}{q_{i,j}} \cdot \beta_j\right)$ such that $p_{i,j} \in \mathbb{Z}$, $q_{i,j} \in \mathbb{N}$. Let $M = \max\left\{\left|\frac{p_{i,j}}{q_{i,j}}\right| : i \in \{1, \dots, n\} \setminus \mathcal{J}, j \in \{0, \dots, m\}\right\}$, let $\widehat{q} = \prod_{i \in \{1,\dots,n\} \setminus \mathcal{J}} \prod_{j=1}^m q_{i,j}$, let $q = \prod_{i \in \{1,\dots,n\} \setminus \mathcal{J}} q_{i,0}$ and let $k = k_0 \cdot q + 1$. Let $\delta > 0$ be arbitrary. Then $1, (k-1) \cdot k \cdot \frac{\beta_1}{2\pi}, \dots, (k-1) \cdot k \cdot \frac{\beta_m}{2\pi}$ is a linearly independent system over $\mathbb{Q}$, hence by Corollary 2.58 we can find $p \in \mathbb{N}$, $d_1, \dots, d_m \in \mathbb{Z}$ such that $\left|p \cdot (k-1) \cdot k \cdot \frac{\beta_j}{2\pi} - (1-k) \cdot \frac{\beta_j}{2\pi} - d_j\right| < \frac{\delta}{M \cdot m \cdot 2\pi}$ for $j = 1, \dots, m$ and $p, d_1, \dots, d_m$ are multiples of $\widehat{q}$. It follows that $|(p \cdot (k-1) + 1) \cdot k \cdot \beta_j - \beta_j - d_j \cdot 2\pi| < \frac{\delta}{M \cdot m}$. By the choice of q and k the numbers defined by $D_i = (p \cdot (k-1) + 1) \cdot k \cdot \frac{p_{i,0}}{q_{i,0}} - \frac{p_{i,0}}{q_{i,0}}$ are integers for all $i \in \{1, \dots, n\} \setminus \mathcal{J}$. Thus

$$\left|(p \cdot (k-1) + 1) \cdot k \cdot \alpha_i - \alpha_i - D_i \cdot 2\pi - \sum_{j=1}^m \frac{p_{i,j}}{q_{i,j}} \cdot d_j \cdot 2\pi\right|$$

$$= \left|\sum_{j=1}^m \frac{p_{i,j}}{q_{i,j}} \left((p \cdot (k-1) + 1) \cdot k \cdot \beta_j - \beta_j\right) - \frac{p_{i,j}}{q_{i,j}} \cdot d_j \cdot 2\pi\right|$$

$$\leq \sum_{j=1}^m \left|\frac{p_{i,j}}{q_{i,j}}\right| \cdot |(p \cdot (k-1) + 1) \cdot k \cdot \beta_j - \beta_j - d_j \cdot 2\pi| < \delta$$

for $i \in \{1, \dots, n\} \setminus \mathcal{J}$, and by the choice of $\widehat{q}$ we have that $\sum_{j=1}^m \frac{p_{i,j}}{q_{i,j}} \cdot d_j \in \mathbb{Z}$. So if we set $z = (p \cdot (k-1) + 1)$ then $\begin{bmatrix}1\end{bmatrix}^{k \cdot z} = \begin{bmatrix}1\end{bmatrix}$, $\begin{bmatrix}-1\end{bmatrix}^{k \cdot z} = \begin{bmatrix}-1\end{bmatrix}$, $B(\alpha_i)^{k \cdot z} = B(\alpha_i)$ for all $i \in \mathcal{J}$ and $B(\alpha_i)^{k \cdot z} = B(\gamma_i)$ for all $i \in \{1, \dots, n\} \setminus \mathcal{J}$ for some $\gamma_i \in (0, 2\pi)$ such that $|\gamma_i - \alpha_i| < \delta$.

So we can approximate T by the powers of T^k, hence we can approximate the powers of T by the powers of T^k. Thus the group generated by T^k is dense in the group generated by T. $\square$

Let $0 < l \leq d$ be integers and let $G_{d,l}$ denote the *Grassmann manifold* of l-dimensional linear subspaces of $\mathbb{R}^d$ equipped with the usual topology (see for example [52, Section 3.9]). For $M \in G_{d,l}$ let $M^\perp \in G_{d,(d-l)}$ denote the orthogonal direct complement of M.

Lemma 2.60. *Let $0 < l < d$ be integers, $v \in \mathbb{R}^d$, $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map with* $\operatorname{rank}(L) = l$ *and $\mathcal{T} \subseteq \mathbb{O}_d$ be such that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$. Then there exists $O_0 \in \overline{\mathcal{T}}$ such that $L \circ O_0(v) = 0$.*

Proof. It is easy to see that if there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ then $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ for every $M \in G_{d,l}$. Let $M \in G_{d,l}$ be such that M is contained in the orthogonal complement of v. Since $\operatorname{rank}(L) = l$ it follows that $\dim(\operatorname{Ker}(L)) = d - l$. Hence $\dim\left(\operatorname{Ker}(L)^\perp\right) = l$. There exists $O_0 \in \overline{\mathcal{T}}$ such that $O_0(M) = \operatorname{Ker}(L)^\perp$. Since v is orthogonal to M it follows that $O_0(v)$ is orthogonal to $O_0(M) = \operatorname{Ker}(L)^\perp$. Thus $O_0(v) \in \operatorname{Ker}(L)$, so $L \circ O_0(v) = 0$. □

We need the following lemma for Example 4.27.

Lemma 2.61. *There exist $T_1, T_2 \in \mathbb{SO}_3$ such that the group generated by T_1 and T_2 is dense in $\mathbb{SO}_3$.*

Proof. Let $\gamma, \delta \notin \mathbb{Q}$ and let

$$T_1 = \begin{pmatrix} \cos(\gamma\pi) & -\sin(\gamma\pi) & 0 \\ \sin(\gamma\pi) & \cos(\gamma\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\delta\pi) & -\sin(\delta\pi) \\ 0 & \sin(\delta\pi) & \cos(\delta\pi) \end{pmatrix}.$$

Let $\mathcal{G}$ be the closure of the group generated by T_1 and T_2. Then $\mathcal{G}$ contains every rotation of the form

$$R_3(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } R_1(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{R}$ by Theorem 2.57. Let $e_1 = (1, 0, 0)^\intercal$. The rotation of the form

$$T \circ R_1(\beta) \circ T^{-1}$$

is the rotation around $T(e_1)$ by angle β. Since every element of $\mathbb{SO}_3$ is a rotation around some line the statement follows if we show that for every $x \in \mathbb{R}^3$, $\|x\| = 1$ there exists $T \in \mathcal{G}$ such that $T(e_1) = x$. Let $x \in \mathbb{R}^3$, $\|x\| = 1$, $x = (a, b, c)^\intercal$. Choose $\alpha, \beta \in \mathbb{R}$ such that $\cos(\alpha) = a$, $\sin(\alpha)\cos(\beta) = b$ and $\sin(\alpha)\sin(\beta) = c$. Then $R_1(\beta)R_3(\alpha)e_1 = x$ and hence the statement follows. □

3 Self-similar sets with no separation condition

While studying self-similar sets the open set condition is a convenient assumption that makes the proofs significantly simpler or sometimes statements do not even hold if the open set condition is not satisfied. That is why self-similar sets satisfying the open set condition are quite well-understood but we know much less in the general situation when no separation condition is assumed. Recent results of Hochman were a major breakthrough in studying overlapping self-similar sets. A folklore conjecture is the following:

Conjecture 3.1. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}$ with attractor K. Then one of the following holds:*
i) There is an exact overlap in $\{S_i\}_{i=1}^m$
ii) $\dim_H(K) = \min\{1, s\}$ where s is the similarity dimension of $\{S_i\}_{i=1}^m$.

One approach which yields results for Lebesgue almost all set in certain family is the transversality technique. This was introduced by Pollicott and Simon [67] however they were building on Falconer's work in [19, 18]. They showed that if we fix the similarity ratios of the maps in an SS-IFS in the line then for almost all translation parameters $\dim_H(K) = \min\{1, s\}$ and Simon and Solomyak [71, Theorem 2.1] showed that if $s > 1$ then for almost all translation parameters $\mathcal{L}^1(K) > 0$. On the other hand, Pollicott and Simon [67, Theorem 1] proved, using the transversality condition, that $\dim_H(K) = \min\{1, s\}$ holds for a certain family of SS-IFS in the line with fixed translation parameters for Lebesgue almost all similarity ratio r in a certain interval, such that every map has the same similarity ratio r. Hochman [39, Theorem 1.5] proves Conjecture 3.1 when only algebraic parameters occur in the defining maps of K. In Example 4.33 we construct a self-similar set $\widehat{K} \subseteq \mathbb{R}$ such that no matter how many times we iterate the IFS then delete any number of exact overlaps such that we do not change the attractor $\widehat{K}$, we still have exact overlaps and hence the similarity dimension never realises the Hausdorff dimension of the set even if changing the defining maps.

We would like to avoid the singular non-interesting case, when K is a single point, which occurs if and only if every S_i has the same fixed point. Hence we make the global assumption throughout the whole study that K contains at least two points. This implies that there are at least two maps in the SS-IFS, i.e. $m > 1$. Hence the similarity dimension of the SS-IFS is strictly positive. It is relevant for us that the assumption that K contains at least two points also implies that K contains infinitely many points and thus $t = \dim_H K > 0$ even with no separation condition by Proposition 2.33.

In this chapter we provide tools for dealing with the overlapping case when no separation condition is assumed. For a self-similar set we find a collection of disjoint cylinder sets that exhaust the set in Hausdorff measure. For such a collection we establish a similarity dimension like formula. This replaces the role of the open set condition in the proofs when $\mathcal{H}^t(K) > 0$. Applying this result we show that $\mathcal{H}^t(A) = \mathcal{H}^t_\infty(A)$ for every Borel set $A \subseteq L(K)$ where $L(K)$ is a linear image of K. As a consequence we deduce that $\mathcal{H}^t(K) > 0$ if and only if $\mathcal{H}^t(K \cap B) \asymp \operatorname{diam}(B)^t$ for every ball B centred in K, i.e. K is Ahlfors regular. We also show that if $t = \dim_H K < 1$ and K is not contained in any affine hyperplane then $\mathcal{H}^t(K) > 0$ is also equivalent to the weak separation property. Thus if $t < 1$ then the weak separation property only depends on K not the SS-IFS.

For Hausdorff dimension we provide a dimension approximation method that allows us to generalise results about non-overlapping self-similar sets to overlapping self-similar sets. We proceed by finding a self-similar set $\widehat{K} \subseteq K$, $\dim_H K - \varepsilon < \dim_H \widehat{K}$ such that $\widehat{K}$ carries convenient properties, for instance $\widehat{K}$ is non-overlapping. Most of the work in this chapter is in [25]. Section 3.2 is a result of collaboration with Fraser [26].

3.1 Treating the lack of separation conditions

The following proposition is a useful tool for generalizing results about Hausdorff dimension known in the case of SSC to the case with no separation condition.

Proposition 3.2. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K. For all $\varepsilon > 0$ there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ that satisfies the SSC, with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K$, $\dim_H K - \varepsilon < \dim_H \widehat{K}$ and the transformation group $\widehat{\mathcal{T}}$ of $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ is dense in $\mathcal{T}$.*

The planar case of Proposition 3.2 was known before and was used, for example, in [64, 61]. The proof in the planar case is not difficult and in three dimensions is not much more complicated. However, higher dimensional cases are more subtle and the proof, given in Section 3.3 , relies on Kronecker's simultaneous approximation theorem (Theorem 2.57).

The following proposition develops a new tool that serves the role of separation conditions in the proofs when the Hausdorff measure of K is positive in its Hausdorff dimension. Note that there exist self-similar sets with positive and finite Hausdorff measure such that they cannot be defined via an SS-IFS satisfying the OSC (see Example 4.33 and Example 4.34). In Section 3.4 we state two other variants of Proposition 3.3 and we hope that such variants of Proposition 3.3 may help to extend other results to settings without any separation condition. The proof of Proposition 3.3, which is given in Section 3.4, relies on Vitali's covering theorem (Theorem 2.16).

Proposition 3.3. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and let $t = \dim_H(K)$. Let $O \in \overline{\mathcal{T}}$ be arbitrary and $\delta > 0$. Then there exists $\mathcal{I}_\infty \subseteq \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that $\|T_{\mathbf{i}} - O\| < \delta$ for all $\mathbf{i} \in \mathcal{I}_\infty$, $K_{\mathbf{i}} \cap K_{\mathbf{j}} = \emptyset$ for $\mathbf{i}, \mathbf{j} \in \mathcal{I}_\infty$, $\mathbf{i} \neq \mathbf{j}$, and $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)\right) = 0$.*

Proposition 3.3 allows us to establish a Moran-Hutchinson like formula for Hausdorff dimension as follows.

Remark 3.4. Under the assumptions of Proposition 3.3 if K is a t-set it follows that $\mathcal{H}^t(K) = \sum_{\mathbf{i}\in\mathcal{I}_\infty} \mathcal{H}^t(K_{\mathbf{i}}) = \sum_{\mathbf{i}\in\mathcal{I}_\infty} r_{\mathbf{i}}^t \cdot \mathcal{H}^t(K)$, hence

$$\sum_{\mathbf{i}\in\mathcal{I}_\infty} r_{\mathbf{i}}^t = 1.$$

This equation plays the role of (2.6) in the non-OSC case.

Another advantage of Proposition 3.3 is that we can regard the IFS as one for which the orthogonal part $T_{\mathbf{i}}$ of the maps are approximately the same at any level. This observation helps us to deal with the higher dimensional cases when the rotations do not necessarily commute.

Proposition 3.3 is proven in Section 3.4.

The next proposition, provided in Section 3.5, says that the Hausdorff measure and content of linear images of K coincide. It follows that the Hausdorff measure is upper semi-continuous since the Hausdorff content is upper semi-continuous. This observation is essential in the proofs of Theorem 4.3 and Theorem 4.5.

Proposition 3.5. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$, let $t = \dim_H(K)$ and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. Then*

(i) $\mathcal{H}^t(L(K)) = \mathcal{H}^t_\infty(L(K))$,

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every linear map $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ with $\|L - L_2\| < \delta$ we have that $\mathcal{H}^t(L_2(K)) \leq \mathcal{H}^t(L(K)) + \varepsilon$.

Semi-continuity of the Hausdorff measure of attractors of SS-IFSs that satisfie the OSC were established by Olsen [60, Theorem 1.1] and semi-continuity of the Lebesgue measure of attractors of more general IFSs were established by Jonker and Veerman [42, Theorem A]. For $M \in G_{d,l}$ let $\Pi_M : \mathbb{R}^d \longrightarrow M$ denote orthogonal projection onto M. In particular, Proposition 3.5 implies that $\mathcal{H}^t(\Pi_M(K))$ is upper semi-continuous in $M \in G_{d,l}$, in contrast with a result of Hochman and Shmerkin [40, Theorem 1.8] on the lower semi-continuity of the lower Hausdorff dimension of Bernoulli convolutions. They show that the 'lower Hausdorff dimension' of Bernoulli convolution is lower semi-continuous in the parameters and we can consider the measures with different parameters as different projections of the same measure. Semi-continuity of the Hausdorff dimension of attractors of more general IFSs were shown by Jonker and Veerman [42, Theorem B].

Proposition 3.5 does not generalise to smooth maps. If K is a 1-set and $L : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a linear map such that $L(K) = [0,1]$ (see Example 4.28 for $t = 1$) then $g(x) := (\cos(L(x), \sin(L(x))) : \mathbb{R}^d \longrightarrow \mathbb{R}^2$ is such that $\mathcal{H}^t(g(K)) \neq \mathcal{H}^t_\infty(g(K))$.

Remark 3.6. It follows by Lemma 2.2 and Proposition 3.5 that $\mathcal{H}^t(B) = \mathcal{H}^t_\infty(B)$ for every $\mathcal{H}^t$-measurable subset $B \subseteq L(K)$. Let $A \subseteq L(K)$ be arbitrary and $B \subseteq L(K)$ be a $\mathcal{H}^t$-measurable hull of A. We can further assume that $\operatorname{diam}(B) = \operatorname{diam}(A)$. Then $\mathcal{H}^t(A) = \mathcal{H}^t(B) = \mathcal{H}^t_\infty(B) \leq \operatorname{diam}(A)^t$. In particular, for $L = Id_{\mathbb{R}^d}$ we obtain that $\mathcal{H}^t(A) \leq \operatorname{diam}(A)^t$ for every subset $A \subseteq K$. Thus Proposition 2.31 remains valid with s replaced by t.

The proof of Proposition 3.5 is provided in Section 3.5.

3.2 Ahlfors regularity and the weak separation property

Proposition 3.5 has applications to the Ahlfors regularity of self-similar sets and related fractals. A bounded set $H \subseteq \mathbb{R}^d$ with Hausdorff dimension t is called *Ahlfors regular* if there exists a constant $c \geq 1$ such that for all $r \in (0, \operatorname{diam}(H)]$ and $x \in H$

$$c^{-1} r^t \leq \mathcal{H}^t\big(H \cap B(x,r)\big) \leq c r^t. \tag{3.1}$$

The following proposition shows that for an Ahlfors regular set the Hausdorff measure and Hausdorff content are equivalent in the Hausdorff dimension (equal up to a constant bound).

Proposition 3.7. *Let $H \subseteq \mathbb{R}^d$ be an Ahlfors regular set with $t = \dim_H H$. Then there exists $c \geq 1$ such that*

$$\mathcal{H}^t_\infty(A) \leq \mathcal{H}^t(A) \leq c\mathcal{H}^t_\infty(A)$$

for every $A \subseteq H$.

Proof. The first inequality $\mathcal{H}^t_\infty(A) \leq \mathcal{H}^t(A)$ always holds by (2.1). Let $A \subseteq H$ and $\{H_n : n \in \mathbb{N}\}$ be a cover of A such that $\operatorname{diam}(H_n) \leq \operatorname{diam}(H)$ for all n, and for all n let $x_n \in H \cap H_n$ if $H \cap H_n \neq \emptyset$ and let $x_n \in H$ be arbitrary if $H \cap H_n = \emptyset$. Then $H_n \subseteq B(x_n, 2\operatorname{diam}(H_n))$ for all n and thus $\{B(x_n, 2\operatorname{diam}(H_n)) : n \in \mathbb{N}\}$ is a cover of A. Let $c \geq 1$ be the constant provided by (3.1). Then

$$\sum_{n\in\mathbb{N}} 2^t \cdot \operatorname{diam}(H_n)^t \geq \sum_{n\in\mathbb{N}} c^{-1}\mathcal{H}^t\big(H \cap B(x_n, 2\operatorname{diam}(H_n))\big) \geq c^{-1}\mathcal{H}^t(A).$$

Taking the infimum over all such covers of A we get that $\mathcal{H}^t_{\operatorname{diam}(H)}(A) \geq 2^{-t}c^{-1}\mathcal{H}^t(A)$. On the other hand, $\mathcal{H}^t_{\operatorname{diam}(H)}(A) = \mathcal{H}^t_\infty(A)$ because $\operatorname{diam}(A) \leq \operatorname{diam}(H)$. Hence $\mathcal{H}^t(A) \leq 2^t c\mathcal{H}^t_\infty(A)$ as required. □

Lemma 3.8. *Let K be a self-similar set. Then there exists $c > 0$ such that for every $x \in K$ and $r \in (0, \operatorname{diam}(K))$ there exists $\mathbf{i} \in \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $K_\mathbf{i} \subseteq B(x,r)$ and $cr \leq \operatorname{diam}(K_\mathbf{i}) < r$.*

Proof. Let $c = \min\{r_i : i \in \mathcal{I}\}$ and $r_{\max} = \max\{r_i : i \in \mathcal{I}\}$. For every $k \in \mathbb{N}$ and $\mathbf{i} \in \mathcal{I}^k$ we have that $\operatorname{diam}(K_\mathbf{i}) = r_\mathbf{i}\operatorname{diam}(K) \leq r_{\max}^k\operatorname{diam}(K)$, so if k is large enough then $\operatorname{diam}(K_\mathbf{i}) < r$ for every $\mathbf{i} \in \mathcal{I}^k$. For every $k \in \mathbb{N}$ by (2.8) there exists $\mathbf{i} \in \mathcal{I}^k$ such that $x \in K_\mathbf{i}$. Thus if $k \in \mathbb{N}$ is sufficiently large then there exists $\mathbf{i} \in \mathcal{I}^k$ with $x \in K_\mathbf{i}$ and $\operatorname{diam}(K_\mathbf{i}) < r$. Let $k \in \mathbb{N}$ be minimal such that there exists $\mathbf{i} \in \mathcal{I}^k$ with $x \in K_\mathbf{i}$ and $\operatorname{diam}(K_\mathbf{i}) < r$. For such $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}^k$ let $\mathbf{j} = (i_1, \dots, i_{k-1})$. It follows by the minimality of k that $\operatorname{diam}(K_\mathbf{j}) \geq r$ because $x \in K_\mathbf{j}$. Thus $r > \operatorname{diam}(K_\mathbf{i}) = \operatorname{diam}(K_\mathbf{j}) \cdot r_k \geq r \cdot c$ and $x \in K_\mathbf{i}$. Since $\operatorname{diam}(K_\mathbf{i}) < r$ and $x \in K_\mathbf{i}$ we have that $K_\mathbf{i} \subseteq B(x,r)$. □

It is well-known that a self-similar set satisfying the open set condition is Ahlfors regular. Proposition 3.5 yields the following Theorem.

Theorem 3.9. *Let K be a self-similar set and $t = \dim_H K$. Then $\mathcal{H}^t(K) > 0$ if and only if K is Ahlfors regular.*

Proof. Obviously when K is is Ahlfors regular then $\mathcal{H}^t(K) > 0$. Now assume that $\mathcal{H}^t(K) > 0$. Let $c > 0$ be the constant provided by Lemma 3.8. Let $x \in K$ and $r \in (0, \operatorname{diam}(K)]$ be arbitrary. Then by Lemma 3.8 there exists $\mathbf{i} \in \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $K_\mathbf{i} \subseteq B(x,r)$ and $cr \leq \operatorname{diam}(K_\mathbf{i}) < r$. Thus by the similarity

$$\mathcal{H}^t(K \cap B(x,r)) \geq \mathcal{H}^t(K_\mathbf{i}) = \operatorname{diam}(K_\mathbf{i})^t \cdot \mathcal{H}^t(K)/\operatorname{diam}(K) \geq (cr)^t\mathcal{H}^t(K)/\operatorname{diam}(K)$$

and by Remark 3.6

$$\mathcal{H}^t(K \cap B(x,r)) \leq (2r)^t,$$

so K is Ahlfors regular. □

Let $K \subseteq \mathbb{R}^d$ be a self-similar set, not contained in any affine hyperplane. Recall that the weak separation property is satisfied if the identity map is an isolated point of the set

$$\left\{ S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k \right\},$$

see (2.7). It was shown in [33, Theorem 2.1] that if K satisfies the weak separation property (which is weaker than the open set condition, see Proposition 2.39) then K is Ahlfors regular. It was also shown [33, Theorem 1.4] that if K does not satisfy the weak separation property then the Assouad dimension $\dim_A K$ of K is greater than or equal to 1. In general, the Assouad dimension is an upper bound for the Hausdorff dimension and we refer the reader to [33] for the definition. This allows us to prove the following corollary.

Corollary 3.10. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$ such that $t = \dim_H K < 1$ and K is not contained in any affine hyperplane. Then the following are equivalent:*

i) $\{S_i\}_{i=1}^m$ satisfies the weak separation property,

ii) $\mathcal{H}^t(F) > 0$,

iii) $0 < \mathcal{H}^t(F) < \infty$,

iv) K is Ahlfors regular,

v) the Hausdorff and Assouad dimensions of K coincide.

Proof. Zerner [76, Corollary after Proposition 2] proved that *i)* $\Rightarrow$ *ii)*, *ii)* and *iii)* are equivalent by Proposition 2.33, Theorem 3.9 shows that *ii)* $\Leftrightarrow$ *iv)*, the fact that *iv)* $\Rightarrow$ *v)* follows by [74, Proposition 2.1 (viii)], and since $\dim_H K < 1$ the result mentioned above [33, Theorem 1.4] shows that *v)* $\Rightarrow$ *i)*. □

We note that Corollary 3.10 also shows that for self-similar sets with Hausdorff dimension strictly less than 1, that are not contained in any affine hyperplane, the weak separation property can be formulated in a way which only depends on the set itself and not the defining iterated function system. The additional assumption $\dim_H F < 1$ required in Corollary 3.10 seems a little strange at first. However, it turns out that this condition is sharp. Firstly consider K in the line.

Example 3.11. The SS-IFS consisting of the maps $S_1(x) = x/2$, $S_2(x) = x/3$ and $S_3(x) = x/2 + 1/2$ does not satisfy the weak separation property. The attractor of this SS-IFS is $K = [0, 1]$, so $\mathcal{H}^1(K) > 0$.

Clearly the attractor is $[0, 1]$. We show that the maps in the form $S_1^{-k} \circ S_2^p = (3^p/2^k)x$ can be arbitrarily close to the identity map, thus the SS-IFS does not satisfy the weak separation property. It is sufficient to show that $3^p/2^k$ can be arbitrarily close to 1 for sufficient $p, k \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary, then $0 < \log(1 + \varepsilon) < \varepsilon$. Since $\log 3/\log 2 \notin \mathbb{Q}$ by Theorem 2.57 we can find $p, k \in \mathbb{N}$ such that $k < p \log 3/\log 2 < k + \log(1 + \varepsilon)$. Thus

$$\log 2^k < \log 3^p < \log 2^k + \frac{\log(1+\varepsilon)}{k} \leq \log 2^k + \log(1 + \varepsilon),$$

hence

$$2^k < 3^p < 2^k(1 + \varepsilon),$$

and so

$$1 < \frac{3^p}{2^k} < 1 + \varepsilon.$$

We use a variation of this example to prove the following proposition demonstrating the (almost) sharpness of Corollary 3.10.

Proposition 3.12. *For all $d \in \mathbb{N} \setminus \{1\}$ and all $t \in (1, d]$, there exists an an SS-IFS $\{S_i\}_{i=1}^m$ with attractor $K \subseteq \mathbb{R}^d$, such that K is not contained in any affine hyperplane and*
i) $\{S_i\}_{i=1}^m$ *does not satisfy the weak separation property,*
ii) $\dim_{\mathrm{H}} K = t$*,*
iii) $\mathcal{H}^t(K) > 0$*.*

Proof. Let $r \in (0, 1/2]$ be chosen such that

$$\frac{\log 2}{-\log r} = \frac{t-1}{d-1} =: u$$

and let $F = [0, 1]$ be viewed as a self-similar attractor of an iterated function system which fails the weak separation property and all of the maps have contraction ratio r. Such an iterated function system can be constructed by modifying [2, Section 2 (v)] where they provide such an example with similarity ratio $r = 1/4$ but the same argument works for arbitryry $r \in (0, 1/2]$. Also, let $E \subseteq [0, 1]$ be the self-similar set defined by the maps $x \mapsto rx$ and $x \mapsto rx + (1 - r)$, and observe that $\dim_{\mathrm{H}} E = u$ and $\mathcal{H}^u(E) > 0$ since the OSC is satisfied. Now let $K = F \times E^{d-1} \subseteq [0, 1]^d$ be the product of F with $d - 1$ copies of E. It is easy to see that K is not contained in any affine hyperplane and that it is a self-similar set defined via the natural product iterated function system. It follows from Theorem 2.29 and Proposition 2.33 that $\dim_H K = 1 + (d - 1)u = t$ and that $\mathcal{H}^t(F) > 0$. Finally it is easy to see that the weak separation property fails by virtue of it failing in the first coordinate. □

For $t = d$ in the above proposition our set K is just $[0, 1]^d$, which is not very interesting. We point out that it is possible to construct a set with the desired properties but which has empty interior. For example, it was shown in [10] that there exists a self-similar set in the plane with positive $\mathcal{H}^2$ measure, but empty interior, and by [76, Theorem 3] such a set must fail to satisfy the weak separation property.

Corollary 3.10 has the following consequence concerning Conjecture 3.1.

Corollary 3.13. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}$ with attractor K, let $\dim_H(K) = t$ and assume that $\mathcal{H}^t(K) > 0$. Then one of the following holds:*
i) There is an exact overlap in $\{S_i\}_{i=1}^m$
ii) $\dim_H(K) = \min\{1, s\}$ *where s is the similarity dimension of* $\{S_i\}_{i=1}^m$*.*

Proof. Assume that *i)* is not satisfied and $t = \dim_H(K) < 1$. Then $\{S_i\}_{i=1}^m$ satisfies the weak separation property by Corollary 3.10. Hence $\{S_i\}_{i=1}^m$ satisfies the OSC by Proposition 2.39 and so $\dim_H(K) = s$ by Theorem 2.34. □

3.3 Dimension approximation of self-similar sets by well behaved self-similar sets

In this section we prove Proposition *3.2*.

Lemma 3.14. *Let $S_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $S_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be contracting similarities with no common fixed point. Then the similarities $S_1^n \circ S_2$ have different fixed points for all $n \in \mathbb{N}$.*

Proof. By Banach's fixed point theorem every contracting similarity $S : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ has a unique fixed point. Assume for a contradiction that there exist $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and a positive integer k such that that $S_1^n \circ S_2(x) = x$ and $S_1^k \circ S_1^n \circ S_2(x) = x$. Then $S_1^{-k}(x) = S_1^n \circ S_2(x) = x$. It follows that the unique fixed point of S_1 is x. But then $S_2(x) = S_1^{-n}(x) = x$ contradicting that S_1 and S_2 have no common fixed point. □

Lemma 3.15. *Let $S_1, \dots, S_m : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be contracting similarities ($m \geq 2$) such that S_1 and S_2 have no common fixed point. Then there exist $F_1, \dots, F_m : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that*

i.) $F_1 = S_1$, $F_2 = S_2$,
ii.) for each $i \in \{3, \dots, m\}$ either $F_i = S_1^{k_i} \circ S_i$ or $F_i = S_2^{k_i} \circ S_i$ for some $k_i \in \mathbb{N}$,
iii.) F_i and F_j have no common fixed point for all $i, j \in \{1, \dots, m\}$, $i \neq j$.

Proof. We prove this by induction on m. If $m = 2$ then it is trivial. Let $m > 2$. Then by the inductive assumption we can find such a system $F_1, \dots, F_{m-1}$ that satisfies the conclusion for $S_1, \dots, S_{m-1}$. The unique fixed point of S_m is either not the fixed point of S_1 or not the fixed point of S_2. Without loss of generality we can assume that S_m and S_1 have no common fixed points. Then by Lemma 3.14 there exists $k_m \in \mathbb{N}$ such that the fixed point of $S_1^{k_m} \circ S_m$ is different from the fixed points of $F_1, \dots, F_{m-1}$. If we set $F_m = S_1^{k_m} \circ S_m$ then $F_1, \dots, F_m$ satisfies the conclusion. □

Now we are ready to prove Proposition *3.2*. The proof consists of two steps. First we find a collection of words $\mathbf{j}_i \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that the group generated by $T_{\mathbf{j}_i}$ is dense in $\mathcal{T}$ and the $S_{\mathbf{j}_i}(K)$ are disjoint. At this point we do not care about the dimension. Then we add another finite set of maps to the new SS-IFS so that the strong separation condition still holds and the dimension becomes arbitrarily close to that of K.

Proof of Proposition 3.2. Since K has at least two points there exist $i, j \in \mathcal{I}$ such that S_i and S_j have no common fixed point, otherwise the common fixed point would be the attractor. Without the loss of generality we can assume that $i = 1$ and $j = 2$.

It follows from Lemma 3.15 that there exist $\mathbf{i}_1, \dots, \mathbf{i}_m \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that $S_{\mathbf{i}_i}$ and $S_{\mathbf{i}_j}$ have no common fixed point for all $i, j \in \mathcal{I} = \{1, \dots, m\}$, $i \neq j$, $\mathbf{i}_1 = 1$, $\mathbf{i}_2 = 2$ and the group generated by $T_{\mathbf{i}_1}, \dots, T_{\mathbf{i}_m}$ is $\mathcal{T}$. Let x_i be the unique fixed point of $S_{\mathbf{i}_i}$ for all $i \in \mathcal{I}$. Let $d_{\min} = \min\{\|x_i - x_j\| : i, j \in \mathcal{I}, i \neq j\} > 0$, $r_{\max} = \max\{r_i : i \in \mathcal{I}\} < 1$ and $N \in \mathbb{N}$ such that $r_{\max}^N \cdot \operatorname{diam}(K) < \frac{d_{\min}}{2}$. Then $S_{\mathbf{i}_i}^{k_i}(K) \cap S_{\mathbf{i}_j}^{k_j}(K) = \emptyset$ for all $i, j \in \mathcal{I}$, $i \neq j$, $k_i, k_j \in \mathbb{N}$, $k_i, k_j \geq N$.

By Proposition 2.59 for all $i \in \mathcal{I}$ we can find $k_i \in \mathbb{N}$, $k_i \geq N$ such that the group generated by $T_{\mathbf{i}_i}^{k_i}$ is dense in the group generated by $T_{\mathbf{i}_i}$. It follows that the group generated by $T_{\mathbf{i}_1}^{k_1}, \dots, T_{\mathbf{i}_m}^{k_m}$ is dense in $\mathcal{T}$ and $S_{\mathbf{i}_i}^{k_i}(K) \cap S_{\mathbf{i}_j}^{k_j}(K) = \emptyset$ for all $i, j \in \mathcal{I}$, $i \neq j$. Let $\widehat{S}_i = S_{\mathbf{i}_i}^{k_i}$ for all $i \in \mathcal{I}$.

Let $F = \bigcup_{i \in \mathcal{I}} S_{\mathbf{i}_i}^{k_i}(K)$. If $K = F$ then $\left\{\widehat{S}_i\right\}_{i=1}^{m}$ satisfies the SSC with attractor $\widehat{K} = K$ and the proof is complete. So we can assume that $F \subsetneq K$. Let $\mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ be such that $K_{\mathbf{j}} \cap F = \emptyset$. Let $t = \dim_H K = \dim_H K_{\mathbf{j}}$. Since K has at least two points it follows that K has infinitely many points, but by Proposition 2.33 $\mathcal{H}^t(K) < \infty$, thus $t > 0$ and hence without loss of generality we can assume that $t > \varepsilon > 0$. Since $\mathcal{H}^{t-\frac{\varepsilon}{2}}(K_{\mathbf{j}}) = \infty$ we can find $\delta > 0$ such that for any 3δ-cover $\mathcal{U}$ of $K_{\mathbf{j}}$ we have that $\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{t-\frac{\varepsilon}{2}} > 1$. Let $r_{\min} = \min\{r_i : i \in \mathcal{I}\} < 1$ and let $\mathcal{J} = \left\{\mathbf{i} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k : K_{\mathbf{i}} \subseteq K_{\mathbf{j}}, r_{\min}\delta \leq \operatorname{diam}(K_{\mathbf{i}}) < \delta\right\}$. Then $\{K_{\mathbf{i}} : \mathbf{i} \in \mathcal{J}\}$ is a cover of $K_{\mathbf{j}}$. Let $\mathbf{j}_1, \dots, \mathbf{j}_n \in \mathcal{J}$ be such that $K_{\mathbf{j}_1}, \dots, K_{\mathbf{j}_n}$ is a maximal pairwise disjoint sub-collection of $\{K_{\mathbf{i}} : \mathbf{i} \in \mathcal{J}\}$. Let U_j be the δ-neighbourhood of $K_{\mathbf{j}_j}$ for $j \in \{1, \dots, n\}$. By the maximality $\{U_j : j \in \{1, \dots, n\}\}$ is a 3δ-cover of $K_{\mathbf{j}}$. Hence by the choice of δ

$$\sum_{j=1}^{n} (3\delta)^{t-\frac{\varepsilon}{2}} \geq \sum_{j=1}^{n} (\operatorname{diam}(U_j))^{t-\frac{\varepsilon}{2}} > 1.$$

It follows that $n \geq (3\delta)^{-\left(t-\frac{\varepsilon}{2}\right)}$. Let K_0 be the attractor of the SS-IFS $\left\{S_{\mathbf{j}_j}\right\}_{j=1}^{n}$. Then $K_0 \subseteq K$, the SS-IFS $\left\{S_{\mathbf{j}_j}\right\}_{j=1}^{n}$ satisfies the SSC and

$$\dim_H K_0 \geq \frac{\log(\frac{1}{n})}{\log(\frac{r_{\min} \cdot \delta}{\operatorname{diam}(K)})} \geq \frac{-\left(t-\frac{\varepsilon}{2}\right) \cdot \log(3) - \left(t-\frac{\varepsilon}{2}\right) \cdot \log(\delta)}{\log(\operatorname{diam}(K)) - \log(r_{\min}) - \log(\delta)}$$

because the similarity dimension of $\left\{S_{\mathbf{j}_j}\right\}_{j=1}^{n}$ is $\dim_H K_0$ by Theorem 2.34. So, by choosing δ small enough, $\dim_H K_0 > t - \varepsilon$. Let $\widehat{m} = m+n$, $\widehat{S_{m+j}} = S_{\mathbf{j}_j}$ for all $j \in \{1, \dots, n\}$ and $\widehat{K}$ be the attractor of the SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$. Then the transformation group $\widehat{\mathcal{T}}$ of $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ is dense in $\mathcal{T}$, $K_0 \subseteq \widehat{K} \subseteq K$, $\dim_H K - \varepsilon < \dim_H K_0 \leq \dim_H \widehat{K}$ and $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ satisfies the SSC. $\square$

A similar argument to the last step of the proof of Proposition 3.2 was used in the proof of [64, Theorem 2].

3.4 Vitali-like exhaustion results for overlapping self-similar sets

In this section our main goal is to prove Proposition 3.3 which provides an important tool to cope with the later results. Proposition 3.3 is essential in the proofs in Section 4.3 and Section 4.4.1 and plays the role of a separation condition when no separation condition is assumed. First we prove two lemmas that we need for the proof.

Lemma 3.16. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and $t = \dim_H(K)$. Then there exists $\mathcal{J} \subseteq \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that $K_{\mathbf{i}} \cap K_{\mathbf{j}} = \emptyset$ for $\mathbf{i}, \mathbf{j} \in \mathcal{J}$, $\mathbf{i} \neq \mathbf{j}$ and $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{J}} K_{\mathbf{i}}\right)\right) = 0$.*

Proof. $\mathcal{H}^t(K) < \infty$ by Proposition 2.33. Let $\mathcal{A} = \left\{K_{\mathbf{i}} : \mathbf{i} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k\right\}$. Then $\mathcal{A}$ is a Vitali cover of K and hence Proposition 2.17 provides a $\mathcal{J}$ with the required properties. □

Remark 3.17. In Lemma 3.16 for a fixed $\delta > 0$ we can further assume that $\operatorname{diam}(K_{\mathbf{i}}) < \delta$ for every $\mathbf{i} \in \mathcal{J}$ because in the proof we can take $\mathcal{A} = \left\{K_{\mathbf{i}} : \mathbf{i} \in \bigcup_{k=N}^{\infty} \mathcal{I}^k\right\}$ for N large enough.

Lemma 3.18. *Let $O \in \overline{\mathcal{T}}$ and $\delta > 0$. Then for each $O_2 \in \overline{\mathcal{T}}$ there exists $\mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that $\|O_2 \circ T_{\mathbf{j}} - O\| < \delta$.*

Proof. By Lemma 2.56 we can find $\mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that

$$\|O_2 \circ T_{\mathbf{j}} - O\| = \left\|T_{\mathbf{j}} - O_2^{-1} \circ O\right\| < \delta.$$

□

Now we are ready to prove Proposition 3.3. In the proof which relies on Vitali's covering theorems, we exhaust a self-similar set in Hausdorff measure by roughly homothetic, disjoint cylinder sets. We find such a collection of cylinders the following way. At first we only exhaust the set by disjoint cylinders. Then with a recursive construction we find the required family of cylinders. At every step we find more and more members of the final collection of cylinders. At every step we exhaust at least a certain proportion of the remaining set, that is not exhausted by the final family members yet, by rougly homothetic cylinders. After countably many steps we end up with the required exhausting family of cylinders.

Proof of Proposition 3.3. Fix $O \in \overline{\mathcal{T}}$. Since $\overline{\mathcal{T}}$ is compact there exists a finite open $\frac{\delta}{2}$-cover $\{U_i\}_{i=1}^q$ of $\overline{\mathcal{T}}$. Let $\mathcal{V} = \{1, \ldots, q\}$ and for every $i \in \mathcal{V}$ fix $O_i \in U_i \bigcap \overline{\mathcal{T}}$. By virtue of Lemma 3.18, for each $i \in \mathcal{V}$ we can find $\mathbf{j}_i \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ such that $\left\|O_i \circ T_{\mathbf{j}_i} - O\right\| < \frac{\delta}{2}$. So for every $\mathbf{i} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k$ there exists $i \in \mathcal{V}$ such that $T_{\mathbf{i}} \in U_i$. Then $\|T_{\mathbf{i}} - O_i\| < \frac{\delta}{2}$ and $\left\|O_i \circ T_{\mathbf{j}_i} - O\right\| < \frac{\delta}{2}$, thus

$$\left\|T_{\mathbf{i}*\mathbf{j}_i} - O\right\| < \delta. \tag{3.2}$$

By Proposition 2.33 $\mathcal{H}^t(K) < \infty$. Let $\mathcal{J} \subseteq \bigcup_{k=1}^{\infty} \mathcal{I}^k$ be the set provided by Lemma 3.16. We define a sequence of sets $\mathcal{I}_1, \mathcal{I}_2, \ldots \subseteq \bigcup_{k=1}^{\infty} \mathcal{I}^k$ inductively. Let $\mathcal{I}_1 = \mathcal{J}$. Given $\mathcal{I}_n$ has been defined we define $\mathcal{I}_{n+1}$ as follows. For each $\mathbf{i} \in \mathcal{I}_n$ we define a set $\mathcal{I}_{n+1,\mathbf{i}}$. If $\|T_{\mathbf{i}} - O\| < \delta$ then let $\mathcal{I}_{n+1,\mathbf{i}} = \{\mathbf{i}\}$. If $\|T_{\mathbf{i}} - O\| \geq \delta$ then $T_{\mathbf{i}} \in U_i$ for some $i \in \mathcal{V}$ and $\left\{K_{\mathbf{i}*\mathbf{j}} : \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k, K_{\mathbf{i}*\mathbf{j}} \cap K_{\mathbf{i}*\mathbf{j}_i} = \emptyset\right\}$ is a Vitali cover of $K_{\mathbf{i}} \setminus K_{\mathbf{i}*\mathbf{j}_i}$, hence by Proposition 2.17 there exists $\mathcal{J}_{n+1,\mathbf{i}} \subseteq \left\{\mathbf{j} : \mathbf{j} \in \bigcup_{k=1}^{\infty} \mathcal{I}^k, K_{\mathbf{i}*\mathbf{j}} \cap K_{\mathbf{i}*\mathbf{j}_i} = \emptyset\right\}$ such that $K_{\mathbf{i}*\mathbf{i}_1} \cap K_{\mathbf{i}*\mathbf{i}_2} = \emptyset$ for $\mathbf{i}_1, \mathbf{i}_2 \in \mathcal{J}_{n+1,\mathbf{i}}$, $\mathbf{i}_1 \neq \mathbf{i}_2$, and $\mathcal{H}^t\left(\left(K_{\mathbf{i}} \setminus K_{\mathbf{i}*\mathbf{j}_i}\right) \setminus \left(\bigcup_{\mathbf{j}\in\mathcal{J}_{n+1,\mathbf{i}}} K_{\mathbf{i}*\mathbf{j}}\right)\right) = 0$. Then let $\mathcal{I}_{n+1,\mathbf{i}} = \{\mathbf{i} * \mathbf{j}_i\} \bigcup \{\mathbf{i} * \mathbf{j} : \mathbf{j} \in \mathcal{J}_{n+1,\mathbf{i}}\}$ and let $\mathcal{I}_{n+1} = \bigcup_{\mathbf{i}\in\mathcal{I}_n} \mathcal{I}_{n+1,\mathbf{i}}$.

Now we define $\mathcal{I}_\infty = \bigcap_{n_1=1}^{\infty}\bigcup_{n_2=n_1}^{\infty}\mathcal{I}_{n_2}$. Clearly $K_{\mathbf{i}} \cap K_{\mathbf{j}} = \emptyset$ for $\mathbf{i},\mathbf{j} \in \mathcal{I}_\infty$, $\mathbf{i} \neq \mathbf{j}$. If $\mathbf{i} \in \mathcal{I}_n$ and $\|T_{\mathbf{i}} - O\| \geq \delta$ then $\mathbf{i} \notin \mathcal{I}_{n+l}$ for every positive integer l, hence $\mathbf{i} \notin \mathcal{I}_\infty$. So $\|T_{\mathbf{i}} - O\| < \delta$ for all $\mathbf{i} \in \mathcal{I}_\infty$. Let $r_{\min} = \min\left\{r_{\mathbf{j}_i} : i \in \mathcal{V}\right\} > 0$. Clearly

$$\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_n} K_{\mathbf{i}}\right)\right) = 0 \tag{3.3}$$

for every positive integer n. For $\mathbf{i} \in \mathcal{I}_n$ such that $\|T_{\mathbf{i}} - O\| \geq \delta$ and $T_{\mathbf{i}} \in U_i$ for some $i \in \mathcal{V}$ (if there are more than one such i then we choose the one that was used above to define the set $\mathcal{J}_{n+1,\mathbf{i}}$) we have that $\{\mathbf{j} : \mathbf{i} * \mathbf{j} \in \mathcal{I}_{n+1}, \|T_{\mathbf{i}*\mathbf{j}} - O\| \geq \delta\} \subseteq \mathcal{J}_{n+1,\mathbf{i}}$ and $\mathcal{H}^t(K_{\mathbf{i}*\mathbf{j}_i}) = r_{\mathbf{j}_i}^t\mathcal{H}^t(K_{\mathbf{i}}) \geq r_{\min}^t\mathcal{H}^t(K_{\mathbf{i}})$, and in the mean time $\left\|T_{\mathbf{i}*\mathbf{j}_i} - O\right\| < \delta$ by (3.2). Therefore $\mathcal{I}_{n+1} \setminus \mathcal{I}_\infty \subseteq \bigcup_{\mathbf{i}\in\mathcal{I}_n\setminus\mathcal{I}_\infty}\{\mathbf{i} * \mathbf{j} : \mathbf{j} \in \mathcal{J}_{n+1,\mathbf{i}}\}$ and

$$\begin{aligned}\mathcal{H}^t\left(\bigcup_{\mathbf{i}\in\mathcal{I}_{n+1}\setminus\mathcal{I}_\infty} K_{\mathbf{i}}\right) &\leq \sum_{\mathbf{i}\in\mathcal{I}_n\setminus\mathcal{I}_\infty}\sum_{\mathbf{j}\in\mathcal{J}_{n+1,\mathbf{i}}}\mathcal{H}^t(K_{\mathbf{i}*\mathbf{j}}) \leq \sum_{\mathbf{i}\in\mathcal{I}_n\setminus\mathcal{I}_\infty}\left(\mathcal{H}^t(K_{\mathbf{i}}) - r_{\min}^t\mathcal{H}^t(K_{\mathbf{i}})\right)\\ &= \sum_{\mathbf{i}\in\mathcal{I}_n\setminus\mathcal{I}_\infty}(1 - r_{\min}^t)\cdot\mathcal{H}^t(K_{\mathbf{i}}) = (1 - r_{\min}^t)\cdot\mathcal{H}^t\left(\bigcup_{\mathbf{i}\in\mathcal{I}_n\setminus\mathcal{I}_\infty} K_{\mathbf{i}}\right).\end{aligned}$$

Hence $\mathcal{H}^t\left(\bigcup_{\mathbf{i}\in\mathcal{I}_{n+1}\setminus\mathcal{I}_\infty} K_{\mathbf{i}}\right) \leq (1 - r_{\min}^t)^n \cdot \mathcal{H}^t\left(\bigcup_{\mathbf{i}\in\mathcal{I}_1\setminus\mathcal{I}_\infty} K_{\mathbf{i}}\right)$ for all $n \in \mathbb{N}$ and combined with (3.3) we get that $\mathcal{H}^t\left(\bigcup_{\mathbf{i}\in\mathcal{I}_{n+1}\bigcap\mathcal{I}_\infty} K_{\mathbf{i}}\right) \geq (1 - (1 - r_{\min}^t)^n)\cdot\mathcal{H}^t(K)$. Thus $\mathcal{H}^t\left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right) \geq \mathcal{H}^t(K)$ and so $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)\right) = 0$. □

Corollary 3.19. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and let $t = \dim_H(K)$. Assume that $\mathcal{T}$ is a finite group and let $O \in \mathcal{T}$ be arbitrary. Then there exists $\mathcal{I}_\infty \subseteq \bigcup_{k=1}^{\infty}\mathcal{I}^k$ such that $T_{\mathbf{i}} = O$ for all $\mathbf{i} \in \mathcal{I}_\infty$, $K_{\mathbf{i}} \cap K_{\mathbf{j}} = \emptyset$ for $\mathbf{i},\mathbf{j} \in \mathcal{I}_\infty$, $\mathbf{i} \neq \mathbf{j}$ and $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)\right) = 0$.*

Proof. Let $q = |\mathcal{T}|$, $\mathcal{T} = \{O_1, \ldots, O_q\}$, $\mathcal{V} = \{1, \ldots, q\}$ and $O = O_i$ for some $i \in \mathcal{V}$. Then let $\delta = \min_{j\in\mathcal{V}, j\neq i}\|O_j - O\| > 0$. By Proposition 3.3 there is an $\mathcal{I}_\infty$ such that $\|T_{\mathbf{i}} - O\| < \delta$ and so $T_{\mathbf{i}} = O$ for all $\mathbf{i} \in \mathcal{I}_\infty$. □

Proposition 3.20. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K and let $t = \dim_H(K)$. Let $O \in \overline{\mathcal{T}}$ be arbitrary and let $\mathbf{i}_1, \ldots, \mathbf{i}_n \in \bigcup_{k=1}^{\infty}\mathcal{I}^k$ be such that $\bigcup_{i=1}^{n} K_{\mathbf{i}_i}$ is a disjoint union and let $\delta > 0$. Then there exists $\mathcal{I}_\infty \subseteq \bigcup_{k=1}^{\infty}\mathcal{I}^k$ such that $\|T_{\mathbf{i}} - O\| < \delta$ for all $\mathbf{i} \in \mathcal{I}_\infty$, with $\bigcup_{\mathbf{i}\in\mathcal{I}_\infty}\bigcup_{i=1}^{n} K_{\mathbf{i}*\mathbf{i}_i}$ a disjoint union and $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty}\bigcup_{i=1}^{n} K_{\mathbf{i}*\mathbf{i}_i}\right)\right) = 0$.*

The proof of Proposition 3.20 is similar to the proof of Proposition 3.3 with the difference that if we have $\mathbf{i} \in \mathcal{I}_n$ at a level such that $\|T_{\mathbf{i}} - O\| < \delta$ then we keep the pieces $K_{\mathbf{i}*\mathbf{i}_i}$ from the next level on and again cover the rest of $K_{\mathbf{i}}$ on the next level.

Remark 3.21. With a slight modification in the proof one can show that Proposition 3.20 remains true even if $Q = \bigcup_{i=1}^{n} K_{\mathbf{i}_i}$ is not necessarily a disjoint union, with the difference that now we only claim $\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} S_{\mathbf{i}}(Q)$ to be a disjoint union for a sufficiante $\mathcal{I}_\infty$.

3.5 Semi-continuity of the Hausdorff measure

First we prove Proposition 3.5 that says the Hausdorff measure of linear images of K is upper semi-continuous in the linear maps. This observation is essential in the proof of Theorem 4.3.

Proposition 3.22. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$, let $t = \dim_H(K)$ and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every linear map $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ with $\|L - L_2\| < \delta$ we have that $\mathcal{H}^t(L_2(K)) \leq \mathcal{H}^t_\infty(L(K)) + \varepsilon$.*

Proof. It is enough to verify the proposition for $0 < \varepsilon < 1$. We may assume that $\mathcal{H}^t(K) > 0$ otherwise $\mathcal{H}^t(L_2(K)) = 0$. By Proposition 2.33 $\mathcal{H}^t(K) < \infty$ hence K is a t-set. Since K is compact there exists $R > 0$ such that K is contained in $B(0, R)$. Since $L(K)$ is compact it follows that $\mathcal{H}^t_\infty(L(K)) < \infty$. Thus there exists a finite open cover $\mathcal{C}$ of $L(K)$ such that

$$\sum_{C \in \mathcal{C}} \operatorname{diam}(C)^t \leq \mathcal{H}^t_\infty(L(K)) + \varepsilon \tag{3.4}$$

and $0 < \operatorname{diam}(C) < \infty$. Let $d_{\max} = \max\{\operatorname{diam}(C) : C \in \mathcal{C}\}$. Because $\mathcal{C}$ is a cover of $L(K)$ and $K \subseteq B(0, R)$ it follows that $K \subseteq \bigcup_{C \in \mathcal{C}} L^{-1}(C) \cap B(0, R)$ and $L^{-1}(C) \cap B(0, R)$ is a bounded set such that $\operatorname{diam}\left(L\left(L^{-1}(C) \cap B(0, R)\right)\right) = \operatorname{diam}(C) > 0$ for each $C \in \mathcal{C}$. Hence for each $C \in \mathcal{C}$ we can find $\delta_C > 0$ such that if $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ is a linear map with $\|L - L_2\| < \delta_C$ then $\operatorname{diam}\left(L_2\left(L^{-1}(C) \cap B(0, R)\right)\right) \leq \operatorname{diam}(C) \cdot (1 + \varepsilon)$. Let $\delta = \min\{\delta_C : C \in \mathcal{C}\} / \|L\| + 1 > 0$. So if $\|L - L_2\| < \delta$ for some linear map $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ and $\|T - Id_{\mathbb{R}^d}\| < \delta$ for some $T \in \mathbb{O}_d$ then

$$\|L - L_2 \circ T\| = \|L - L \circ T + L \circ T - L_2 \circ T\| < \delta(\|L\| + 1) = \min\{\delta_C : C \in \mathcal{C}\}.$$

Hence

$$\operatorname{diam}\left(L_2 \circ T\left(L^{-1}(C) \cap B(0, R)\right)\right) \leq \operatorname{diam}(C) \cdot (1 + \varepsilon) < 2d_{\max}. \tag{3.5}$$

The lemma will follow if we show that $\mathcal{H}^t_\eta(L_2(K)) \leq (1 + \varepsilon)^t \cdot (\mathcal{H}^t_\infty(L(K)) + \varepsilon)$ for every $\eta > 0$ and every linear map $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ with $\|L - L_2\| < \delta$. Let $\eta > 0$ be fixed, $r_{\max} = \max\{r_i : i \in \mathcal{I}\}$, let k be a positive integer such that $r^k_{\max} \cdot 2d_{\max} < \eta$ and $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map with $\|L - L_2\| < \delta$. Then K is the attractor of the SS-IFS $\left\{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$. We apply Lemma 3.3 to the SS-IFS $\left\{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ with $O = Id_{\mathbb{R}^d}$ and $\delta > 0$ to obtain $\mathcal{I}_\infty \subseteq \bigcup_{k_2=1}^\infty \mathcal{I}^{k \cdot k_2}$ such that $\|T_{\mathbf{i}} - Id_{\mathbb{R}^d}\| < \delta$ for all $\mathbf{i} \in \mathcal{I}_\infty$, $K_{\mathbf{i}} \cap K_{\mathbf{j}} = \emptyset$ for $\mathbf{i}, \mathbf{j} \in \mathcal{I}_\infty$, $\mathbf{i} \neq \mathbf{j}$, and $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{I}_\infty} K_{\mathbf{i}}\right)\right) = 0$. As K is a t-set we have by Remark 3.4 that

$$\sum_{\mathbf{i} \in \mathcal{I}_\infty} r^t_{\mathbf{i}} = 1. \tag{3.6}$$

Since $\|T_{\mathbf{i}} - Id_{\mathbb{R}^d}\| < \delta$ for $\mathbf{i} \in \mathcal{I}_\infty$ and $\|L - L_2\| < \delta$, it follows from (3.5) that

$$\begin{aligned}\operatorname{diam}\left(L_2\left(S_{\mathbf{i}}\left(L^{-1}(C) \cap B(0, R)\right)\right)\right) &\leq r_{\mathbf{i}} \cdot \operatorname{diam}\left(L_2\left(T_{\mathbf{i}}\left(L^{-1}(C) \cap B(0, R)\right)\right)\right) \\ &\leq r_{\mathbf{i}} \cdot \operatorname{diam}(C) \cdot (1 + \varepsilon) < r_{\mathbf{i}} \cdot 2d_{\max} \\ &\leq r^k_{\max} \cdot 2d_{\max} < \eta.\end{aligned} \tag{3.7}$$

Thus $\{L_2(S_{\mathbf{i}}(L^{-1}(C)\cap B(0,R))) : C\in\mathcal{C}, \mathbf{i}\in\mathcal{I}_\infty\}$ is an η-cover of $L_2\left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)$ and

$$\begin{aligned}\mathcal{H}^t_\eta\left(L_2\left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)\right) &\le \sum_{C\in\mathcal{C}}\sum_{\mathbf{i}\in\mathcal{I}_\infty} \mathrm{diam}\left(L_2\left(S_{\mathbf{i}}\left(L^{-1}(C)\cap B(0,R)\right)\right)\right)^t \\ &\le \sum_{C\in\mathcal{C}}\sum_{\mathbf{i}\in\mathcal{I}_\infty} \left(r_{\mathbf{i}}\cdot \mathrm{diam}\,(C)\cdot(1+\varepsilon)\right)^t \\ &\le (1+\varepsilon)^t\sum_{C\in\mathcal{C}}\mathrm{diam}\,(C)^t\sum_{\mathbf{i}\in\mathcal{I}_\infty} r_{\mathbf{i}}^t \le (1+\varepsilon)^t\sum_{C\in\mathcal{C}}\mathrm{diam}\,(C)^t \\ &\le (1+\varepsilon)^t\cdot\left(\mathcal{H}^t_\infty(L(K))+\varepsilon\right)\end{aligned} \tag{3.8}$$

where we used (3.4), (3.6) and (3.7).

Since $\mathcal{H}^t\left(K\setminus\left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)\right)=0$ it follows $\mathcal{H}^t_\eta\left(L_2\left(K\setminus\left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} K_{\mathbf{i}}\right)\right)\right)=0$. Thus by (3.8)

$$\mathcal{H}^t_\eta(L_2(K))\le(1+\varepsilon)^t\cdot\left(\mathcal{H}^t_\infty(L(K))+\varepsilon\right)$$

which completes the proof. □

Proof of Proposition 3.5. Because $\mathcal{H}^t_\infty(L(K))\le\mathcal{H}^t(L(K))$ by (2.1) statement *ii)* is an immediate consequence of Proposition 3.22. Taking $L_2=L$ in Proposition 3.22 as ε approaches 0 we get statement *i)*. □

4 Projections of self-similar sets

The results of this chapter are mostly contained in [25].

Studying the Hausdorff dimension and measure of orthogonal projections and linear images of sets has a long history. The most fundamental result is that for an analytic subset K of $\mathbb{R}^d$

$$\dim_H \Pi_M(K) = \min\{l, \dim_H(K)\}$$

for almost all l-dimensional subspaces M, where $\Pi_M : \mathbb{R}^d \longrightarrow M$ denotes orthogonal projection onto M. If $\dim_H(K) > l$ then

$$\mathcal{H}^l(\Pi_M(K)) > 0$$

for almost all l-dimensional subspaces M. These results were proved in the case $d = 2$, $l = 1$ by Marstrand [51], and generalized to higher dimensions by Mattila [53]. If l is an integer then we call an l-set K *irregular* if $\mathcal{H}^l(K \cap M) = 0$ for every differentiable l-manifold M. It was shown by Besicovitch [6] in the planar case and by Federer [28] in the higher dimensional cases that for an l-set K where l is an integer

$$\mathcal{H}^l(\Pi_M(K)) = 0$$

for almost all l-dimensional subspaces M if and only if K is irregular. If K is not irregular then $\mathcal{H}^l(\Pi_M(K)) > 0$ for almost all l-dimensional subspaces M.

While the results above provide information about generic projections they do not give any information about an individual projection or linear image of the set. There are examples that show that the 'exceptional set' for which the conclusions do not hold can be 'big' [51]. Analyzing the image of a set under a particular linear map is more difficult even in simple cases, see for example Kenyon [44] and Hochman [39, Theorem 1.6] who consider the 1-dimensional Sierpinski gasket.

It is easy to see that if K is a self-similar set with all the defining maps homotheties then every linear image of K is itself a self-similar set. It was asked by Mattila [54, Problem 2] in the planar case 'what can be said about the measures $\mathcal{H}^t(\Pi_M(K))$ if $t = \dim_H(K) < 1$ and the defining maps contain rotations?'. Eroğlu [13] showed that if the open set condition is satisfied and the orthogonal part of one of the defining maps is a rotation of infinite order then $\mathcal{H}^t(\Pi_M(K)) = 0$ for every line M. We generalize this result to higher dimensions and for continuously differentiable maps in place of projections without assuming any separation condition. We obtain results on the structure of linear images of K if the transformation group generated by the orthogonal parts of the defining maps is of finite order. We show that linear images of such self-similar sets are graph directed self-similar sets. We establish an invariance result concerning the Hausdorff measure of the linear images of K in the general case with no restrictions on the orthogonal transformation group. As a consequence of this we conclude that for every linear map into another Euclidean space $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ where d_2 is an arbitrary natural number and for disjoint subsets A and B of K we have that $\mathcal{H}^t\left(L(A) \cap L(B)\right) = 0$ even if no separation condition is satisfied. In particular, the projections of disjoint parts of K are almost disjoint.

Peres and Shmerkin [64, Theorem 5] showed that if the orthogonal part of one of the defining maps is a rotation of infinite order then

$$\dim_H \Pi_M(K) = \min\{1, \dim_H(K)\}$$

for every line M. Very recently Hochman and Shmerkin [40, Corollary 1.7] generalized this to higher dimensions for continuously differentiable maps in the strong separation condition case. Using their result and a dimension approximation method we deduce the same conclusion without any separation condition. On the other hand, we show that if the orthogonal transformation group generated by the orthogonal parts of the defining maps is of finite order then there exists a projection of K such that the dimension drops under the image of the projection.

4.1 Linear images of self-similar sets

It is well-known that if K is an attractor of an SS-IFS such that $|\mathcal{T}| = 1$ then $\Pi_M(K)$ is also a self-similar set for every l-dimensional subspace M. It was shown by Fraser [32, Lemma 2.7] that the vertical and horizontal projections of certain 'box-like' planar self-affine sets are graph directed attractors. We show that, in the case of finite $\mathcal{T}$, similar results can be obtained on the structure of the linear images of self-similar sets.

Theorem 4.1. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$, of similarity dimension s and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. Assume that $\mathcal{T} = \{O_1, \ldots, O_q\}$ is a finite group where $q = |\mathcal{T}|$. Then there exists a strongly connected GD-IFS in $\mathbb{R}^{d_2}$ with attractor $(L \circ O_1(K), \ldots, L \circ O_q(K))$ such that s is the similarity dimension of this GD-IFS with T_e the identity map for all directed edges e, and additionally $\mathcal{H}^s(L \circ O_1(K)) = \ldots = \mathcal{H}^s(L \circ O_q(K))$.*

Our next result states that if the Hausdorff dimension of K equals its similarity dimension and $\mathcal{T}$ is finite then we can always find a projection such that the dimension drops under the projection. We show this by finding a projection where exact overlapping occurs. We note that the assumption, that the Hausdorff and the similarity dimensions are the same, is weaker than the OSC, see [66, Theorem 1.1].

Theorem 4.2. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$ of similarity dimension s. Assume that $\mathcal{T}$ is finite and let $l \in \mathbb{N}$, $l < d$. Then there exists an l-dimensional subspace $M \subseteq \mathbb{R}^d$ such that $\dim_H(\Pi_M(K)) < s$.*

The proofs of Theorem 4.1 and Theorem 4.2 are given in Section 4.2.

The result of Theorem 4.1, that $\mathcal{H}^s(L \circ O_i(K)) = \mathcal{H}^s(L \circ O_j(K))$, suggests the following theorem.

Theorem 4.3. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$, let $t = \dim_H(K)$ and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map. If $\mathcal{H}^t(K) > 0$ then*

$$\mathcal{H}^t(L \circ O(A)) = \frac{\mathcal{H}^t(A)}{\mathcal{H}^t(K)} \mathcal{H}^t(L(K)) \tag{4.1}$$

for all $A \subseteq K$ and $O \in \overline{\mathcal{T}}$.

We note that the assumption in Theorem 4.3, that $\mathcal{H}^t(K) > 0$, is again a weaker condition than the OSC (see Example 4.33 and Example 4.34) and the only role of this assumption is that we can divide by $\mathcal{H}^t(K)$ in the formula. If $\mathcal{H}^t(K) = 0$ then $\mathcal{H}^t(L(K)) = 0$ for every linear map L. In Example 4.34 we construct a self-similar set K with $0 < \mathcal{H}^t(K) < \infty$ such that there exists no SS-IFS with attractor K that satisfies the OSC.

Theorem 4.3 has an interesting consequence, that the linear images of disjoint parts of K are 'almost disjoint' even if no separation condition is satisfied.

Corollary 4.4. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor $K \subseteq \mathbb{R}^d$, let $t = \dim_H(K)$, let $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map and $A, B \subseteq K$ be such that $\mathcal{H}^t(A \cap B) = 0$ and A is $\mathcal{H}^t$-measurable. Then $\mathcal{H}^t(L(A) \cap L(B)) = 0$.*

Theorem4.3 and Corollary 4.4 are shown in Section 4.3.

In [13] Eroğlu showed that if the transformation group of an SS-IFS in $\mathbb{R}^2$ contains a dense set of rotations in $\mathbb{SO}_2$ then $\mathcal{H}^s(\Pi_M(K)) = 0$ for all lines M, where s denotes the similarity dimension of the SS-IFS. Eroğlu's result does not give any information about the projections when the OSC is not satisfied. Using a different approach we generalize this result to higher dimensions for differentiable maps in place of projections and without any separation condition, with s replaced by the Hausdorff dimension of K. Both Eroğlu's and our proof are based on the idea of finding two or several cylinders and fixed direction in which their projection have large overlap. Eroğlu uses similar arguments to those of Simon and Solomyak [72] to show that the projection measure has infinite upper density almost everywhere, whilst we use a Vitali covering argument to show that the Hausdorff measure of the image must collapse. The proof is provided in Section 4.4.1.

Theorem 4.5. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K, let $t = \dim_H(K)$, let U be an open neighbourhood of K and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. Then $\mathcal{H}^t(g(K)) = 0$ for every continuously differentiable map $g : U \longrightarrow \mathbb{R}^{d_2}$ such that $\operatorname{rank}(g'(x)) \leq l$ for every $x \in K$.*

If $\operatorname{rank}(g'(x)) = d$ for some $x \in K$ then g is a bi-Lipschitz function between a neigbourhood V of x and $g(V)$ and hence $\mathcal{H}^t(g(K)) = 0$ if and only if $\mathcal{H}^t(K) = 0$.

We note that the assumption, that $\{O(M) : O \in \mathcal{T}\}$ is dense for some $M \in G_{d,l}$, is equivalent to $\{O(M) : O \in \mathcal{T}\}$ being dense for each $M \in G_{d,l}$.

It was shown by Peres and Shmerkin [64, Theorem 5] on the plane under the conditions of Theorem 4.5 that $\dim_H(\Pi_M(K)) = \min\{t, 1\}$ for every line M. This was generalized to higher dimensions by Hochman and Shmerkin [40, Corollary 1.7] for SS-IFS that satisfies the SSC and the SSC was relaxed by Falconer and Jin [22, Corollary 5.2] to the 'strong variational principle'. We use the result of Hochman and Shmerkin and a dimension approximation method (Proposition 3.2) to deduce the same conclusion without any separation condition.

Theorem 4.6. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K, let $t = \dim_H(K)$, let U be an open neighbourhood of K and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. Then $\dim_H(g(K)) = \min\{t, l\}$ for every continuously differentiable map $g : U \longrightarrow \mathbb{R}^l$ such that $\operatorname{rank}(g'(x)) = l$ for some $x \in K$.*

We state a corollary of Theorem 4.6 which applies to $g : U \longrightarrow \mathbb{R}^{d_2}$ where d_2 may be greater than m.

Corollary 4.7. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K, let $t = \dim_H(K)$, let U be an open neighbourhood of K and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. If $g : U \longrightarrow \mathbb{R}^{d_2}$ is a continuously differentiable map such that $\operatorname{rank}(g'(x)) = l$ for every $x \in K$ and either of the following conditions is satisfied*

(i) $g \in C^\infty$,

(ii) $t \leq l$

then $\dim_H (g(K)) = \min\{t, l\}$.

The proofs of Theorem 4.6 and Corollary 4.7 are provided in Section 4.4.2.

In the planar case $|\mathcal{T}| = \infty$ is equivalent to $\{O(M) : O \in \mathcal{T}\}$ being dense in $G_{2,1}$ for every $M \in G_{2,1}$. Furthermore, it can be easily shown that in the planar case $|\mathcal{T}| = \infty$ also implies that $\mathcal{T}$ contains a rotation of infinite order. Hence the following result on the plane is an immediate consequence of Theorem 4.5 and Theorem 4.6

Corollary 4.8. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^2$ with attractor K, let $t = \dim_H(K)$, let $g : U \longrightarrow \mathbb{R}$ be a continuously differentiable map where U is an open neighbourhood of K and assume that $|\mathcal{T}| = \infty$. Then $\mathcal{H}^t(g(K)) = 0$. If $g' \neq 0$ for some $x \in K$ then $\dim_H (g(K)) = \min\{t, 1\}$.*

Example 4.30 shows that in general $|\mathcal{T}| = \infty$ does not imply either the conclusion of Theorem 4.6 or the conclusion of Theorem 4.5 in higher dimensions.

Example 4.32 shows that neither the conclusion of Theorem 4.6 nor the conclusion of Theorem 4.5 necessarily remain true if we replace g with a Lipschitz function that is a composition of an orthogonal projection and a bi-Lipschitz map.

In [35] Furstenberg introduces the definition of a 'dimension conserving map'. If $f : A \longrightarrow \mathbb{R}^{d_2}$ is a Lipschitz map where $A \subseteq \mathbb{R}^d$ we say that f is dimension conserving if, for some $\delta \geq 0$,

$$\delta + \dim_H \left\{ y \in f(A) : \dim_H(f^{-1}(y)) \geq \delta \right\} \geq \dim_H A \tag{4.2}$$

with that convention that $\dim_H(\emptyset) = -\infty$ so that δ cannot be chosen too large. Furstenberg also introduces 'mini- and micro-sets of a set', and a compact set is defined to be 'homogeneous' if all of its micro-sets are also mini-sets. Furstenberg's main theorem [35, Theorem 6.2] states that the restriction of a linear map to a homogeneous compact set is dimension conserving. He suggests that if K is a self-similar set, $\mathcal{T}$ has only one element and the SSC is satisfied then K is homogeneous. One can show that K is homogeneous even if $\mathcal{T}$ is finite and the SSC is satisfied. Thus for such K the restriction of any linear map to K is dimension conserving even though, by Theorem 4.2, there must be a projection under which the dimension drops. Theorem 4.6 implies that if $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ where $\dim_H K \leq l$, then the restriction of g to K is dimension conserving, where g is a continuously differentiable map of rank l.

4.2 Iterated function systems with finite transformation groups

In this section we deal with the case when $\mathcal{T}$ is finite. First, using a natural construction of a GD-IFS we verify Theorem 4.1. Then we prove Theorem 4.2.

Proof of Theorem 4.1. We need to construct a directed graph $G(\mathcal{V}, \mathcal{E})$ and a GD-IFS $\{S_e : e \in \mathcal{E}\}$ that satisfies the theorem. Let $\mathcal{V}$ be the set $\{1, 2, \ldots, q\}$. For $i, j \in \mathcal{V}$ and for $n \in \mathcal{I}$ we draw a directed edge $e_{i,j}^n$ from i to j if $O_i \circ T_n = O_j$. Then let $\mathcal{E} = \{e_{i,j}^n : i, j \in \mathcal{V}, n \in \mathcal{I}, O_i \circ T_n = O_j\}$.

For $i, j \in \mathcal{V}$ and $n \in \mathcal{I}$ such that $O_i \circ T_n = O_j$, i.e. $e_{i,j}^n = e \in \mathcal{E}$, we write $v_e = v_{e_{i,j}^n} = L \circ O_i(v_n)$, $r_e = r_{e_{i,j}^n} = r_n$ and let $S_e : \mathbb{R}^{d_2} \longrightarrow \mathbb{R}^{d_2}$ be the map

$$S_e(x) = S_{e_{i,j}^n}(x) = r_e \cdot x + v_e \tag{4.3}$$

Let $\{S_e : e \in \mathcal{E}\}$ be the GD-IFS on the graph $G(\mathcal{V}, \mathcal{E})$. Since $K = \bigcup_{n=1}^m S_n(K)$, for $i \in \mathcal{V}$,

$$\begin{aligned} L(O_i(K)) &= \bigcup_{n=1}^m L \circ O_i \circ S_n(K) = \bigcup_{n=1}^m (r_n \cdot L \circ O_i \circ T_n(K) + L \circ O_i(v_n)) \\ &= \bigcup_{n=1}^m \bigcup_{j \in \mathcal{V}, O_i \circ T_n = O_j} (r_n \cdot L \circ O_j(K) + L \circ O_i(v_n)) = \bigcup_{j=1}^q \bigcup_{e \in \mathcal{E}_{i,j}} S_e(L \circ O_j(K)) \end{aligned}$$

and this shows that the q-tuple $(L \circ O_1(K), \ldots, L \circ O_q(K))$ is the attractor of $\{S_e : e \in \mathcal{E}\}$.

Let us show that the graph $G(\mathcal{V}, \mathcal{E})$ is strongly connected. Let $i, j \in \mathcal{V}$ be arbitrary. Then $O_i^{-1} \circ O_j \in \mathcal{T}$ and since $\mathcal{T}$ is generated by the transformations $\{T_i\}_{i=1}^m$ and each T_i has finite order there exists $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ such that $T_{\mathbf{i}} = T_{i_1} \circ \ldots \circ T_{i_k} = O_i^{-1} \circ O_j$. Let $\mathbf{j} = (j_1, \ldots, j_k, j_{k+1}) \in \mathcal{V}^{k+1}$ be such that $j_1 = i$ and $O_{j_n} \circ T_{i_n} = O_{j_{n+1}}$ for $1 \le n \le k$. This shows that there exists a k step directed path from i to j, that visits vertices $i = j_1, \ldots, j_k, j_{k+1} = j$ in order. So the graph $G(\mathcal{V}, \mathcal{E})$ is strongly connected.

Let $u = (1, \ldots, 1)^{\intercal} \in \mathbb{R}^q$ be the vector with each coordinate 1. For the GD-IFS $\{S_e : e \in \mathcal{E}\}$ the matrix $A^{(s)}$ is defined as in (2.11) $A_{i,j}^{(s)} = \sum_{e \in \mathcal{E}_{i,j}} r_e^s$, hence the ith coordinate of the vector $A^{(s)}u$ is

$$\left(A^{(s)}u\right)_i = \sum_{j=1}^q A_{i,j}^{(s)} = \sum_{j=1}^q \sum_{e \in \mathcal{E}_{i,j}} r_e^s = \sum_{j=1}^q \sum_{n=1}^m \sum_{e_{i,j}^n \in \mathcal{E}_{i,j}} r_{e_{i,j}^n}^s = \sum_{n=1}^m r_n^s = 1$$

using that s is the similarity dimension of the SS-IFS $\{S_i\}_{i=1}^m$. So u is a non-negative, non-zero eigenvector of the irreducible matrix $A^{(s)}$ with eigenvalue 1. Thus $\rho(A^{(s)}) = 1$ by Corollary 2.46 and hence the similarity dimension of the GD-IFS $\{S_e : e \in \mathcal{E}\}$ is s.

Let $y_i = \mathcal{H}^s(L \circ O_i(K))$ and $y^{\intercal} = (y_1, \ldots, y_q)$. If the GD-IFS $\{S_e : e \in \mathcal{E}\}$ does not satisfy the OSC then $\mathcal{H}^s(L \circ O_1(K)) = \ldots = \mathcal{H}^s(L \circ O_q(K)) = 0$ by Theorem 2.43. If the GD-IFS $\{S_e : e \in \mathcal{E}\}$ satisfies the OSC then $0 < \mathcal{H}^s(L \circ O_i(K)) < \infty$ for each $i \in \mathcal{V}$ by Theorem 2.43, hence $y \in \mathbb{R}^q$, $y > 0$ and by Remark 2.45 $A^{(s)}y = y$. So y is a positive scalar multiple of u by Theorem 2.44 *(iv)*. So $y = \mathcal{H}^s(L \circ O_i(K)) \cdot u$ for each $i \in \mathcal{V}$ and hence $\mathcal{H}^s(L \circ O_1(K)) = \ldots = \mathcal{H}^s(L \circ O_q(K))$. $\square$

Theorem 4.2 states that we can always find a projection such that the dimension drops under the image of the projection. We show this by finding a projection where exact overlapping occurs.

Proof of Theorem 4.2. We can assume that $l = d-1$ because if $M \in G_{d,d-1}$ such that $\dim_H(\Pi_M(K)) < s$ and N is a subspace contained in M then $\Pi_N = \Pi_N \circ \Pi_M$ hence $\dim_H(\Pi_N(K)) < s$. Let $\mathcal{T} = \{O_1, \ldots, O_q\}$ where $q = |\mathcal{T}|$ and let $\mathcal{V} = \{1, 2, \ldots, q\}$. We may assume that $T_1 = T_2 = Id_{\mathbb{R}^d}$ because if we iterate the IFS q times then we obtain the SS-IFS $\{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^q\}$. The similarity dimension of this new SS-IFS is s, the attractor of it is K and the transformation group of it is a subgroup of $\mathcal{T}$, hence is finite. Since q is the order of $\mathcal{T}$ it follows that $T_1^q = T_2^q = Id_{\mathbb{R}^d}$. So taking the new IFS, after relabeling, we have that $T_1 = T_2 = Id_{\mathbb{R}^d}$. We can further assume that $r_1 = r_2$ because if we iterate the IFS, we obtain the SS-IFS $\{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^2\}$ and again the similarity dimension, the attractor and the finiteness of the transformation group do not change. Then $r_1 \cdot r_2 = r_2 \cdot r_1$, $T_1 \circ T_2 = Id_{\mathbb{R}^d} \circ Id_{\mathbb{R}^d} = T_2 \circ T_1$. So taking the new IFS, after relabeling, we have that $T_1 = T_2 = Id_{\mathbb{R}^d}$ and $r_1 = r_2$.

So $K_1 = S_1(K)$ is a translate of $K_2 = S_2(K)$. Let v be the translation vector such that $K_1 = K_2 + v$. Let M be the orthogonal direct complement of v (if $v = 0$ then $M \in G_{d,d-1}$ can be arbitrary). Then $\Pi_M(K_1) = \Pi_M(K_2)$. Let $L = \Pi_M : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$. Then let $G(\mathcal{V}, \mathcal{E})$ be the graph, $\{S_e : e \in \mathcal{E}\}$ be the GD-IFS that is constructed in the proof of Theorem 4.1, and for $i, j \in \mathcal{V}$ and for $n \in \mathcal{I}$ such that $O_i \circ T_n = O_j$, let $e_{i,j}^n$ be as in the proof. Let $i \in \mathcal{V}$ be such that $O_i = Id_{\mathbb{R}^d}$. Then $e_{i,i}^1$ and $e_{i,i}^2$ are loops in $G(\mathcal{V}, \mathcal{E})$ and

$$\begin{aligned} S_{e_{i,i}^1}(\Pi_M(K)) &= r_1 \cdot \Pi_M(K) + \Pi_M(v_1) = r_1 \cdot \Pi_M(K) + \Pi_M(v_1 - v) \\ &= r_2 \cdot \Pi_M(K) + \Pi_M(v_2) = S_{e_{i,i}^2}(\Pi_M(K)). \end{aligned}$$

So if we take $\{S_e : e \in \mathcal{E} \setminus \{e_0\}\}$ with $e_0 = e_{i,i}^2$ then $\{S_e : e \in \mathcal{E} \setminus \{e_0\}\}$ is a strongly connected GD-IFS with attractor $(\Pi_M \circ O_1(K), \ldots, \Pi_M \circ O_q(K))$. So by Lemma 2.48 the similarity dimension of $\{S_e : e \in \mathcal{E} \setminus \{e_0\}\}$ is strictly smaller than s. Hence $\dim_H(\Pi_M(K)) < s$ by Lemma 2.40. $\square$

4.3 Hausdorff measure of the orbits

In this section we deal with the general results when we have no restriction on $\mathcal{T}$. Our main aim is to prove Theorem 4.3. At the end of this section we conclude Corollary 4.4 from Theorem 4.3.

Proof of Theorem 4.3. By Proposition 2.33, $\mathcal{H}^t(K) < \infty$, hence K is a t-set as $\mathcal{H}^t(K) > 0$ by assumption. Let $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ be such that for every linear map $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ with $\|L - L_2\| < \delta$ we have that $\mathcal{H}^t(L_2(K)) \leq \mathcal{H}^t(L(K)) + \varepsilon$. Such a $\delta > 0$ exists by Proposition 3.5. Let $\mathcal{I}_\infty$ be the set provided by Proposition 3.3 for O^{-1} in place of O and $\frac{\delta}{\|L\|}$ in place of δ. Then $\|O \circ T_{\mathbf{i}} - Id_{\mathbb{R}^d}\| = \|T_{\mathbf{i}} - O^{-1}\| < \frac{\delta}{\|L\|}$ for every $\mathbf{i} \in \mathcal{I}_\infty$, hence $\|L \circ O \circ T_{\mathbf{i}} - L\| \leq \|L\| \cdot \|O \circ T_{\mathbf{i}} - Id_{\mathbb{R}^d}\| < \delta$. So $\mathcal{H}^t(L \circ O \circ T_{\mathbf{i}}(K)) \leq \mathcal{H}^t(L(K)) + \varepsilon$ for every $\mathbf{i} \in \mathcal{I}_\infty$, hence

$$\mathcal{H}^t\left(L \circ O(S_{\mathbf{i}}(K))\right) = r_{\mathbf{i}}^t \cdot \mathcal{H}^t\left(L \circ O \circ T_{\mathbf{i}}(K)\right) \leq r_{\mathbf{i}}^t \cdot \left(\mathcal{H}^t\left(L(K)\right) + \varepsilon\right). \tag{4.4}$$

Since K is a t-set we have by Remark 3.4 that

$$\sum_{\mathbf{i} \in \mathcal{I}_\infty} r_{\mathbf{i}}^t = 1. \tag{4.5}$$

It follows that

$$\sum_{\mathbf{i} \in \mathcal{I}_\infty} \mathcal{H}^t\left(L \circ O(S_{\mathbf{i}}(K))\right) \leq \sum_{\mathbf{i} \in \mathcal{I}_\infty} r_{\mathbf{i}}^t \cdot \left(\mathcal{H}^t\left(L(K)\right) + \varepsilon\right) = \mathcal{H}^t\left(L(K)\right) + \varepsilon$$

where we have used (4.4) and (4.5). Because $\mathcal{H}^t\left(L \circ O\left(K \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{I}_\infty} S_{\mathbf{i}}(K)\right)\right)\right) = 0$ it follows that $\mathcal{H}^t\left(L \circ O(K)\right) \leq \mathcal{H}^t\left(L(K)\right) + \varepsilon$ for all $\varepsilon > 0$. Hence $\mathcal{H}^t\left(L \circ O(K)\right) \leq \mathcal{H}^t\left(L(K)\right)$. Replacing L by $L \circ O$ and O by O^{-1}, with the same argument we get that $\mathcal{H}^t\left(L(K)\right) = \mathcal{H}^t\left(L \circ O \circ O^{-1}(K)\right) \leq \mathcal{H}^t\left(L \circ O(K)\right)$. Thus $\mathcal{H}^t\left(L \circ O(K)\right) = \mathcal{H}^t\left(L(K)\right)$ and so (4.1) holds for $A = K$.

Let $\mathbf{i} \in \mathcal{I}^k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{H}^t\left(L \circ O(K_{\mathbf{i}})\right) &= \mathcal{H}^t\left(L \circ O(S_{\mathbf{i}}(K))\right) = \mathcal{H}^t\left(L \circ O(r_{\mathbf{i}} \cdot T_{\mathbf{i}}(K) + v_{\mathbf{i}})\right) \\ &= r_{\mathbf{i}}^t \cdot \mathcal{H}^t\left(L \circ O \circ T_{\mathbf{i}}(K)\right) = \frac{\mathcal{H}^t(K_{\mathbf{i}})}{\mathcal{H}^t(K)} \cdot \mathcal{H}^t\left(L(K)\right)\end{aligned}$$

where we used (4.1) when $A = K$. So (4.1) holds for $A = K_{\mathbf{i}}$, for each $\mathbf{i} \in \mathcal{I}^k$, $k \in \mathbb{N}$.

Let $\mathcal{J}$ be the set provided by Lemma 3.16. For every $k \in \mathbb{N}$ let us denote the set $\{\mathbf{i}_1 * \ldots * \mathbf{i}_k : \mathbf{i}_1, \ldots, \mathbf{i}_k \in \mathcal{J}\}$ by $\mathcal{J}^k$. For $k \in \mathbb{N}$

$$\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{J}^k} K_{\mathbf{i}}\right)\right) = 0, \tag{4.6}$$

thus

$$\mathcal{H}^t(K) = \mathcal{H}^t\left(\bigcup_{\mathbf{i} \in \mathcal{J}^k} K_{\mathbf{i}}\right) = \sum_{\mathbf{i} \in \mathcal{J}^k} \mathcal{H}^t(K_{\mathbf{i}}).$$

So

$$\begin{aligned}\sum_{\mathbf{i} \in \mathcal{J}^k} \mathcal{H}^t\left(L \circ O(K_{\mathbf{i}})\right) &= \sum_{\mathbf{i} \in \mathcal{J}^k} \frac{\mathcal{H}^t(K_{\mathbf{i}})}{\mathcal{H}^t(K)} \cdot \mathcal{H}^t\left(L(K)\right) = \frac{\mathcal{H}^t(K)}{\mathcal{H}^t(K)} \cdot \mathcal{H}^t\left(L(K)\right) \\ &= \mathcal{H}^t\left(L \circ O(K)\right) = \mathcal{H}^t\left(L \circ O\left(\bigcup_{\mathbf{i} \in \mathcal{J}^k} K_{\mathbf{i}}\right)\right)\end{aligned}$$

where we used (4.1) for $A = K_{\mathbf{i}}$ and for $A = K$. It follows that $\mathcal{H}^t\left(L \circ O(K_{\mathbf{i}} \cap K_{\mathbf{j}})\right) = 0$ for $\mathbf{i}, \mathbf{j} \in \mathcal{J}^k$, $\mathbf{i} \neq \mathbf{j}$. Hence (4.1) holds for $A = \bigcup_{\mathbf{i} \in \mathcal{F}} K_{\mathbf{i}}$ where $\mathcal{F} \subseteq \mathcal{J}^k$.

Using (4.6) and the countable subadditivity of measures it follows that

$$\mathcal{H}^t\left(K\setminus\left(\bigcap_{k=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{J}^k}K_{\mathbf{i}}\right)\right)=0. \tag{4.7}$$

Assume that $A\subseteq\bigcap_{k=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{J}^k}K_{\mathbf{i}}$ is compact, $\varepsilon>0$ arbitrary and let

$$F_k=\bigcup_{\mathbf{i}\in\mathcal{J}^k,K_{\mathbf{i}}\cap A\neq\emptyset}K_{\mathbf{i}}.$$

Then $K\supseteq F_1\supseteq F_2\supseteq\ldots$ and $A=\bigcap_{k=1}^{\infty}F_k$. Thus there exists k such that $\mathcal{H}^t\left(F_k\setminus A\right)<\varepsilon$. Since (4.1) holds for F_k it follows that

$$\begin{aligned}\mathcal{H}^t\left(L\circ O(A)\right)\leq\mathcal{H}^t\left(L\circ O(F_k)\right)&=\frac{\mathcal{H}^t(F_k)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)\\&=\frac{\mathcal{H}^t(A)+\mathcal{H}^t(F_k\setminus A)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)\leq\frac{\mathcal{H}^t(A)+\varepsilon}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\mathcal{H}^t\left(L\circ O(A)\right)&\geq\mathcal{H}^t\left(L\circ O(F_k)\right)-\mathcal{H}^t\left(L\circ O(F_k\setminus A)\right)\\&\geq\frac{\mathcal{H}^t(F_k)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)-\left\|L\circ O\right\|^t\cdot\mathcal{H}^t\left(F_k\setminus A\right)\\&\geq\frac{\mathcal{H}^t(A)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)-\left\|L\right\|^t\cdot\varepsilon.\end{aligned} \tag{4.9}$$

Since $\varepsilon>0$ is arbitrary (4.1) holds for compact $A\subseteq\bigcap_{k=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{J}^k}K_{\mathbf{i}}$.

Now assume that A is any $\mathcal{H}^t$-measurable set and $\varepsilon>0$ is arbitrary. By (4.7)

$$\mathcal{H}^t\left(A\bigcap\left(\bigcap_{k=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{J}^k}K_{\mathbf{i}}\right)\right)=\mathcal{H}^t(A).$$

Hence we can find a compact $F\subseteq A\bigcap\left(\bigcap_{k=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{J}^k}K_{\mathbf{i}}\right)\subseteq A$ such that $\mathcal{H}^t(A\setminus F)<\varepsilon$. Since (4.1) holds for F it follows that

$$\begin{aligned}\mathcal{H}^t\left(L\circ O(A)\right)\geq\mathcal{H}^t\left(L\circ O(F)\right)&=\frac{\mathcal{H}^t(F)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)\\&=\frac{\mathcal{H}^t(A)-\mathcal{H}^t(A\setminus F)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)\geq\frac{\mathcal{H}^t(A)-\varepsilon}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\mathcal{H}^t\left(L\circ O(A)\right)&\leq\mathcal{H}^t\left(L\circ O(F)\right)+\mathcal{H}^t\left(L\circ O(A\setminus F)\right)\\&\leq\frac{\mathcal{H}^t(F)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)+\left\|L\circ O\right\|^t\cdot\mathcal{H}^t\left(A\setminus F\right)\\&\leq\frac{\mathcal{H}^t(A)}{\mathcal{H}^t(K)}\mathcal{H}^t\left(L(K)\right)+\left\|L\right\|^t\cdot\varepsilon.\end{aligned} \tag{4.11}$$

Since $\varepsilon > 0$ is arbitrary (4.1) holds for every $\mathcal{H}^t$-measurable A.

Now let $A \subseteq K$ be arbitrary and let B be a $\mathcal{H}^t$-measurable hull of A such that $A \subseteq B \subseteq K$. By virtue of Lemma 2.4 and applying (4.1) to B we get that (4.1) holds for A. □

Proof of Corollary 4.4. If $\mathcal{H}^t(K) = 0$ then the statement is trivial, so we can assume that $\mathcal{H}^t(K) > 0$. Since $B \subseteq (K \setminus A) \cup (A \cap B)$ and $\mathcal{H}^t(A \cap B) = 0$ it is enough to show that $\mathcal{H}^t(L(A) \cap L(K \setminus A)) = 0$. By Theorem 4.3

$$\begin{aligned}
\mathcal{H}^t(L(K)) &= \mathcal{H}^t(L(A) \cup L(K \setminus A)) \\
&= \mathcal{H}^t(L(A)) + \mathcal{H}^t(L(K \setminus A)) - \mathcal{H}^t(L(A) \cap L(K \setminus A)) \\
&= \frac{\mathcal{H}^t(A)}{\mathcal{H}^t(K)}\mathcal{H}^t(L(K)) + \frac{\mathcal{H}^t(K \setminus A)}{\mathcal{H}^t(K)}\mathcal{H}^t(L(K)) - \mathcal{H}^t(L(A) \cap L(K \setminus A)) \\
&= \mathcal{H}^t(L(K)) - \mathcal{H}^t(L(A) \cap L(K \setminus A)).
\end{aligned}$$

Hence $\mathcal{H}^t(L(A) \cap L(K \setminus A)) = 0$ since $\mathcal{H}^t(L(K)) < \infty$ by Proposition 2.33. □

4.4 Transformation groups of dense orbits

This section deals with the situation where the action of the transformation group $\mathcal{T}$ has a dense orbit in the Grassmann manifold $G_{d,l}$. We split the section into two parts, one that handles the Hausdorff measure images under differentiable mappings and the other that deals with their Hausdorff dimension.

4.4.1 Hausdorff measure

In this section our main goal is to prove Theorem 4.5. First we show that under the assumptions of Theorem 4.5 every linear image of K is of zero measure. Then we generalise this for continuously differentiable maps.

Lemma 4.9. *Let $\mathcal{G}$ be a closed subset of $\mathbb{O}_d$, let $K \subseteq \mathbb{R}^d$ be a compact set, $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map and $c > 0$ be such that $\mathcal{H}^t_\infty(L \circ O(K)) < c$ for every $O \in \mathcal{G}$. Then there exists $\zeta > 0$ such that for every $O \in \mathcal{G}$ there exists a finite open cover $\mathcal{U}$ of $L \circ O(K)$ such that $\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^t < c$ and $\min_{U \in \mathcal{U}} \operatorname{diam}(U) > \zeta$.*

Proof. For every $O \in \mathcal{G}$ we can find a finite open cover $\mathcal{U}_O$ of $L \circ O(K)$ and $0 < \varepsilon_O < \frac{1}{2}$ such that $\sum_{U \in \mathcal{U}_O} \operatorname{diam}(U)^t \cdot (1 + 2\varepsilon_O)^t < c$. Let $\zeta_O = \min_{U \in \mathcal{U}_O} \operatorname{diam}(U) > 0$ and $\widehat{U}$ be the $\zeta_O \cdot \varepsilon_O$-neigbourhood of U for each $U \in \mathcal{U}_O$. We can find $\delta_O > 0$ such that if $O_2 \in \mathbb{O}_d$ and $\|O - O_2\| < \delta_O$ then $L \circ O_2(K)$ is contained in the $\zeta_O \cdot \varepsilon_O$-neighbourhood of $L \circ O(K)$, hence $L \circ O_2(K)$ is covered by $\left\{\widehat{U} : U \in \mathcal{U}_O\right\}$. Then $\left\{\widehat{U} : U \in \mathcal{U}_O\right\}$ is an open cover of $L \circ O_2(K)$,

$$\sum_{U \in \mathcal{U}_O} \operatorname{diam}(\widehat{U})^t \leq \sum_{U \in \mathcal{U}_O} \operatorname{diam}(U)^t \cdot (1 + 2\varepsilon_O)^t < c$$

and $\min_{U \in \mathcal{U}_O} \operatorname{diam}(\widehat{U}) > \zeta_O$.

As $\mathcal{G}$ is compact, we can find finitely many orthogonal transformations $O_1, \dots, O_n \in \mathcal{G}$ such that for every $O \in \mathcal{G}$ there exists $i \in \{1, \dots, n\}$ with $\|O_i - O\| < \delta_{O_i}$. Hence $\zeta = \min_{1 \le i \le n} \zeta_{O_i}$ satisfies the statement. $\square$

Recall that for $r \in \mathbb{R}$, $r > 0$ and $H \subseteq \mathbb{R}^d$ we denote the *r-neigbourhood* of H by $B(H, r)$, i.e. $B(H, r) = \left\{x \in \mathbb{R}^d : \exists y \in H, \|x - y\| < r\right\}$.

Proposition 4.10. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K, let $t = \dim_H(K)$ and $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map with $\operatorname{rank}(L) = l$. If $1 \le l < d$ and there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ then $\mathcal{H}^t\left(L(K)\right) = 0$.*

We first show that there exist two words a and b and $O_0 \in \mathbb{O}_d$ such that $L \circ O_0(K_a)$ and $L \circ O_0(K_b)$ have very large overlap. Then we use a variant of Proposition 3.3 to show that, due to self-similarity, this remains valid at all scales. Finally we conclude that due to these overlaps the measure must collapse.

Proof. It holds in general that $\mathcal{H}^t(H) = 0$ if and only if $\mathcal{H}^t_\infty(H) = 0$. Hence it is enough to show that $\mathcal{H}^t_\infty\left(L(K)\right) = 0$. We can assume that $\mathcal{H}^t\left(K\right) > 0$ otherwise the statement is trivial. By Proposition 2.33 $\mathcal{H}^t\left(K\right) < \infty$, hence K is a t-set. It follows that $\mathcal{H}^t\left(L(K)\right) < \infty$ and by Theorem 4.3 and Proposition 3.5

$$\mathcal{H}^t_\infty\left(L(K)\right) = \mathcal{H}^t\left(L(K)\right) = \mathcal{H}^t\left(L \circ O_0(K)\right) = \mathcal{H}^t_\infty\left(L \circ O_0(K)\right)$$

for every $O_0 \in \overline{\mathcal{T}}$. Let $\varepsilon > 0$ be arbitrary, $\mathcal{G} = \overline{\mathcal{T}}$, $c = \mathcal{H}^t_\infty\left(L(K)\right) + \varepsilon$ and $\zeta > 0$ be the ζ provided by Lemma 4.9. We can find $\delta > 0$ such that for every linear map $L_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\|Id_{\mathbb{R}^d} - L_2\| < \delta$ we have that $L_2(K) \subseteq B\left(K, \varepsilon\zeta\right)$. By Proposition 3.3 we can find $\mathbf{i}_1, \mathbf{i}_2 \in \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $K_{\mathbf{i}_1} \cap K_{\mathbf{i}_2} = \emptyset$ and $\|T_{\mathbf{i}_1} - Id_{\mathbb{R}^d}\| < \frac{\delta}{4}$, $\|T_{\mathbf{i}_2} - Id_{\mathbb{R}^d}\| < \frac{\delta}{4}$. Let $a = \mathbf{i}_1 * \mathbf{i}_2$ and $b = \mathbf{i}_2 * \mathbf{i}_1$. Then $\|T_a - Id_{\mathbb{R}^d}\| < \frac{\delta}{2}$, $\|T_b - Id_{\mathbb{R}^d}\| < \frac{\delta}{2}$, $K_a \cap K_b = \emptyset$ and $r_a = r_b$. Let $v = S_b(0) - S_a(0)$ and $O_0 \in \overline{\mathcal{T}}$ such that $L \circ O_0(v) = 0$. We can choose such an O_0 by Lemma 2.60. We can find $\delta_2 > 0$ such that if $\|L \circ O_0 - L_2\| < \delta_2$ then $\|L_2(v)\| < r_a\varepsilon\zeta$.

We can apply Proposition 3.20 with $\min\left\{\frac{\delta}{2}, \frac{\delta_2}{\|L\|}\right\}$ replacing δ, a replacing $\mathbf{i}_1$, b replacing $\mathbf{i}_2$, $n = 2$ and $O = Id_{\mathbb{R}^d}$ to obtain $\mathcal{I}_\infty \subseteq \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $\|T_{\mathbf{i}} - Id_{\mathbb{R}^d}\| < \min\left\{\frac{\delta}{2}, \frac{\delta_2}{\|L\|}\right\}$ for all $\mathbf{i} \in \mathcal{I}_\infty$, with $\bigcup_{\mathbf{i} \in \mathcal{I}_\infty}\left(K_{\mathbf{i}*a} \cup K_{\mathbf{i}*b}\right)$ a disjoint union and $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{I}_\infty}\left(K_{\mathbf{i}*a} \cup K_{\mathbf{i}*b}\right)\right)\right) = 0$.

So $\|T_{\mathbf{i}} \circ T_a - Id_{\mathbb{R}^d}\| < \delta$ and $\|T_{\mathbf{i}} \circ T_b - Id_{\mathbb{R}^d}\| < \delta$, hence $T_{\mathbf{i}} \circ T_a(K) \subseteq B(K, \varepsilon\zeta)$ and $T_{\mathbf{i}} \circ T_b(K) \subseteq B(K, \varepsilon\zeta)$. Thus

$$r_{\mathbf{i}} r_a T_{\mathbf{i}} \circ T_a(K) \cup r_{\mathbf{i}} r_b T_{\mathbf{i}} \circ T_b(K) \subseteq B(r_{\mathbf{i}} r_a K, r_{\mathbf{i}} r_a \varepsilon\zeta)$$

since $r_a = r_b$. Hence

$$O_0 \circ S_{\mathbf{i}*a}(K) \subseteq B\left(r_{\mathbf{i}} r_a O_0(K) + O_0 \circ S_{\mathbf{i}*a}(0), r_{\mathbf{i}} r_a \varepsilon\zeta\right)$$

and

$$O_0 \circ S_{\mathbf{i}*b}(K) \subseteq B\left(r_{\mathbf{i}} r_a O_0(K) + O_0 \circ S_{\mathbf{i}*b}(0), r_{\mathbf{i}} r_a \varepsilon\zeta\right).$$

Hence

$$L \circ O_0 \circ S_{\mathbf{i}*a}(K) \subseteq B\left(r_{\mathbf{i}} r_a L \circ O_0(K) + L \circ O_0 \circ S_{\mathbf{i}*a}(0), \|L\| r_{\mathbf{i}} r_a \varepsilon\zeta\right)$$

and

$$L \circ O_0 \circ S_{\mathbf{i}*b}(K) \subseteq B\left(r_{\mathbf{i}} r_a L \circ O_0(K) + L \circ O_0 \circ S_{\mathbf{i}*a}(0) + L \circ O_0 \circ r_{\mathbf{i}} T_{\mathbf{i}}(v), \|L\| r_{\mathbf{i}} r_a \varepsilon\zeta\right).$$

By the choice of δ_2 we have $\|L \circ O_0 \circ r_{\mathbf{i}} T_{\mathbf{i}}(v)\| < r_{\mathbf{i}} r_a \varepsilon\zeta$. Hence

$$L \circ O_0 \circ S_{\mathbf{i}*a}(K) \cup L \circ O_0 \circ S_{\mathbf{i}*b}(K) \subseteq B\left(r_{\mathbf{i}} r_a L \circ O_0(K) + L \circ O_0 \circ S_{\mathbf{i}*a}(0), (\|L\|+1) r_{\mathbf{i}} r_a \varepsilon\zeta\right).$$

By the choice of ζ there exists an open cover $\mathcal{U}$ of $L \circ O_0(K)$ such that $\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^t < \mathcal{H}^t_\infty(L(K)) + \varepsilon$ and $\min_{U\in\mathcal{U}} \operatorname{diam}(U) > \zeta$. Let $\widehat{U} = B(U, (\|L\|+1)\varepsilon\zeta)$ for each $U \in \mathcal{U}$ and $\mathcal{A} = \left\{ r_{\mathbf{i}} r_a \widehat{U} + L \circ O_0 \circ S_{\mathbf{i}*a}(0) : U \in \mathcal{U}, \mathbf{i} \in \mathcal{I}_\infty \right\}$. Then $\mathcal{A}$ is an open cover of $L \circ O_0 \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} (K_{\mathbf{i}*a} \cup K_{\mathbf{i}*b}) \right)$.

Because K is a t-set it follows in a similar way to Remark 3.4 that $\sum_{\mathbf{i}\in\mathcal{I}_\infty} r^t_{\mathbf{i}}(r^t_a + r^t_b) = 1$, hence $\sum_{\mathbf{i}\in\mathcal{I}_\infty} r^t_{\mathbf{i}}(r^t_a) = \frac{1}{2}$ because $r_a = r_b$. Thus

$$\begin{aligned}
\sum_{A\in\mathcal{A}} \operatorname{diam}(A)^t &\leq \sum_{U\in\mathcal{U}} \sum_{\mathbf{i}\in\mathcal{I}_\infty} r^t_{\mathbf{i}} r^t_a \left(\operatorname{diam}(U) + 2(\|L\|+1)\varepsilon\zeta\right)^t \\
&\leq \sum_{U\in\mathcal{U}} \sum_{\mathbf{i}\in\mathcal{I}_\infty} r^t_{\mathbf{i}} r^t_a \operatorname{diam}(U)^t \left(1 + 2(\|L\|+1)\varepsilon\right)^t \\
&= \sum_{U\in\mathcal{U}} \frac{1}{2} \operatorname{diam}(U)^t \left(1 + 2(\|L\|+1)\varepsilon\right)^t \\
&\leq \frac{1}{2} \left(\mathcal{H}^t_\infty(L(K)) + \varepsilon\right) \left(1 + 2(\|L\|+1)\varepsilon\right)^t.
\end{aligned}$$

Because $\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} (K_{\mathbf{i}*a} \cup K_{\mathbf{i}*b})\right)\right) = 0$ it follows that

$$\mathcal{H}^t_\infty\left(L \circ O_0\left(K \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty} (K_{\mathbf{i}*a} \cup K_{\mathbf{i}*b})\right)\right)\right) = 0.$$

Hence

$$\mathcal{H}^t_\infty(L \circ O_0(K)) \leq \frac{1}{2} \left(\mathcal{H}^t_\infty(L(K)) + \varepsilon\right) \left(1 + 2(\|L\|+1)\varepsilon\right)^t.$$

Since this is true for all $\varepsilon > 0$ it follows that

$$\mathcal{H}^t_\infty(L(K)) = \mathcal{H}^t_\infty(L \circ O_0(K)) \leq \frac{1}{2} \cdot \mathcal{H}^t_\infty(L(K)).$$

Thus $\mathcal{H}^t_\infty(L(K)) = 0$. □

Corollary 4.11. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS with attractor K, let $t = \dim_H(K)$ and let $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map with $\operatorname{rank}(L) \leq l$. If $1 \leq l < d$ and there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ then $\mathcal{H}^t(L(K)) = 0$.*

Proof. If $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ is a linear map of rank k and $k \leq l < d$ then $\dim \mathrm{Ker}(L) = d - k$. Let $N \in G_{d,d-l}$ such that $N \subseteq \mathrm{Ker}(L)$. Then $L = L \circ \Pi_{N^\perp}$. It follows from Proposition 4.10 that $\mathcal{H}^t(\Pi_{N^\perp}(K)) = 0$. Hence $\mathcal{H}^t(L(K)) = 0$. □

Lemma 4.12. *Let $K \subseteq \mathbb{R}^d$ be a compact set and $c, M > 0$ be constants such that $\mathcal{H}^t_\infty(L(K)) < c$ for every linear map $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ with $\|L\| \leq M$. Then there exists $\zeta > 0$ such that for every linear map $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ with $\|L\| \leq M$ there exists a finite open cover $\mathcal{U}$ of $L(K)$ such that $\sum_{U \in \mathcal{U}} \mathrm{diam}(U)^t < c$ and $\min_{U \in \mathcal{U}} \mathrm{diam}(U) > \zeta$.*

Lemma 4.12 can be proved similarly to Lemma 4.9 due the fact that the unit ball of the set of linear maps between two finite dimensional Euclidean spaces is compact.

Lemma 4.13. *Let $K \subseteq \mathbb{R}^d$ be a compact set, let U be a neighbourhood of K and let $g : U \longrightarrow \mathbb{R}^{d_2}$ be a continuously differentiable map. Then for every $\zeta > 0$ there exists $\delta > 0$ such if $x, y \in K$ with $\|y - x\| < \delta$ then $\|g(y) - g(x) - g'(x) \cdot (y - x)\| \leq \zeta \cdot \|y - x\|$.*

Lemma 4.13 follows from [9, Exercise 7(c).3] and the fact that g is continuously differentiable on the compact set K.

Proof of Theorem 4.5. We can assume that $\mathcal{H}^t(K) > 0$ otherwise the statement is trivial since g is a Lipschitz map. By Proposition 2.33 $\mathcal{H}^t(K) < \infty$ and hence K is a t-set. Let $\varepsilon > 0$ be fixed.

Let $x_0 \in K$ be arbitrary. It follows from Corollary 4.11 that

$$\mathcal{H}^t_\infty(g'(S_{\mathbf{i}}(x_0)) \circ T_{\mathbf{i}}(K)) = \mathcal{H}^t(g'(S_{\mathbf{i}}(x_0)) \circ T_{\mathbf{i}}(K)) = 0.$$

Let $c = \varepsilon$ and $M = \sup\{\|g'(x)\| : x \in K\} < \infty$, then let $\zeta > 0$ be the ζ provided by Lemma 4.12. Hence for every $\mathbf{i} \in \bigcup_{k=1}^\infty \mathcal{I}^k$ there exists a finite open cover $\mathcal{U}_{\mathbf{i}}$ of $g'(S_{\mathbf{i}}(x_0)) \circ T_{\mathbf{i}}(K)$ such that

$$\sum_{U \in \mathcal{U}_{\mathbf{i}}} \mathrm{diam}(U)^t < \varepsilon \tag{4.12}$$

and $\min_{U \in \mathcal{U}_{\mathbf{i}}} \mathrm{diam}(U) > \zeta$.

By Lemma 4.13 we can find $\delta > 0$ such that

$$\left\| g(y) - g(x) - g'(x) \cdot (y - x) \right\| \leq \zeta/2 \cdot \|y - x\|$$

for $x, y \in K$ such that $\|y - x\| < \delta$. By Lemma 3.16 and Remark 3.17 we can find $\mathcal{J} \subseteq \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $K_{\mathbf{i}} \cap K_{\mathbf{j}} = \emptyset$ for $\mathbf{i}, \mathbf{j} \in \mathcal{J}$, $\mathbf{i} \neq \mathbf{j}$ and $\mathrm{diam}(K_{\mathbf{i}}) < \delta$ for every $\mathbf{i} \in \mathcal{J}$ and

$$\mathcal{H}^t\left(K \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{J}} K_{\mathbf{i}}\right)\right) = 0. \tag{4.13}$$

Similarly to Remark 3.4 it follows that

$$\sum_{\mathbf{i} \in \mathcal{J}} r_{\mathbf{i}}^t = 1. \tag{4.14}$$

For every $y \in K$ we have that $\|S_{\mathbf{i}}(y) - S_{\mathbf{i}}(x_0)\| \leq \mathrm{diam}(K_{\mathbf{i}}) < \delta$ and hence by the choice of δ it follows that

$$\left\| g\left(S_{\mathbf{i}}(y)\right) - g\left(S_{\mathbf{i}}(x_0)\right) - g'\left(S_{\mathbf{i}}(x_0)\right) \cdot \left(S_{\mathbf{i}}(y) - S_{\mathbf{i}}(x_0)\right) \right\|$$

$$\leq \zeta/2 \cdot \|S_{\mathbf{i}}(y) - S_{\mathbf{i}}(x_0)\| \leq \zeta/2 r_{\mathbf{i}} \mathrm{diam}(K).$$

Thus

$$g\left(S_{\mathbf{i}}(y)\right) \in B\left(g'\left(S_{\mathbf{i}}(x_0)\right)\left(S_{\mathbf{i}}(K)\right) + g\left(S_{\mathbf{i}}(x_0)\right) - g'\left(S_{\mathbf{i}}(x_0)\right) \cdot \left(S_{\mathbf{i}}(x_0)\right), \zeta r_{\mathbf{i}} \mathrm{diam}(K) \right). \tag{4.15}$$

Since $\mathcal{U}_{\mathbf{i}}$ is an open cover of $g'(S_{\mathbf{i}}(x_0)) \circ T_{\mathbf{i}}(K)$ it follows from (4.15) that

$$\left\{ B\left(r_{\mathbf{i}} U + g'\left(S_{\mathbf{i}}(x_0)\right) \cdot g\left(S_{\mathbf{i}}(0)\right) + g\left(S_{\mathbf{i}}(x_0)\right) - g'\left(S_{\mathbf{i}}(x_0)\right) \cdot \left(S_{\mathbf{i}}(x_0)\right), \zeta r_{\mathbf{i}} \mathrm{diam}(K) \right) : U \in \mathcal{U}_{\mathbf{i}} \right\}$$

is an open cover of $g\left(S_{\mathbf{i}}(K)\right)$. We have that

$$\mathrm{diam}\left(B\left(r_{\mathbf{i}} U + g'\left(S_{\mathbf{i}}(x_0)\right) \cdot g\left(S_{\mathbf{i}}(0)\right) + g\left(S_{\mathbf{i}}(x_0)\right) - g'\left(S_{\mathbf{i}}(x_0)\right) \cdot \left(S_{\mathbf{i}}(x_0)\right), \zeta r_{\mathbf{i}} \mathrm{diam}(K) \right) \right)$$

$$\leq \mathrm{diam}\left(B\left(r_{\mathbf{i}} \cdot U, \zeta r_{\mathbf{i}} \mathrm{diam}(K)\right) \right) \leq r_{\mathbf{i}}\left(\mathrm{diam}(U) + 2\zeta \mathrm{diam}(K)\right) \tag{4.16}$$

$$\leq r_{\mathbf{i}} \mathrm{diam}(U)\left(1 + 2\mathrm{diam}(K)\right).$$

Since g is a Lipschitz map it follows from (4.13) that

$$\mathcal{H}^t_\infty\left(g\left(K \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{J}} K_{\mathbf{i}} \right) \right) \right) = 0.$$

Hence by (4.16)

$$\begin{aligned} \mathcal{H}^t_\infty\left(g(K)\right) &\leq \sum_{\mathbf{i} \in \mathcal{J}} \sum_{U \in \mathcal{U}_{\mathbf{i}}} r_{\mathbf{i}}^t \mathrm{diam}(U)^t \left(1 + 2\mathrm{diam}(K)\right)^t \\ &\leq \left(1 + 2\mathrm{diam}(K)\right)^t \sum_{\mathbf{i} \in \mathcal{J}} r_{\mathbf{i}}^t \sum_{U \in \mathcal{U}_{\mathbf{i}}} \mathrm{diam}(U)^t < \left(1 + 2\mathrm{diam}(K)\right)^t \varepsilon \end{aligned}$$

where we used (4.12) and (4.14). Since this is true for every $\varepsilon > 0$ it follows that $\mathcal{H}^t_\infty\left(g(K)\right) = 0$ and hence $\mathcal{H}^t\left(g(K)\right) = 0$. $\square$

4.4.2 Hausdorff dimension

In this section we prove Theorem 4.6 and Corollary 4.7. We derive Theorem 4.6 from [40, Corollary 1.7] and Proposition 3.2. Finally we conclude Corollary 4.7 from Theorem 4.6.

Proof of Theorem 4.6. The upper bound $\dim_H(g(K)) \leq \min\{t,l\}$ follows since g is a Lipschitz map on K.

First assume that $\operatorname{rank}(g'(x)) = l$ holds for every $x \in U$. By Proposition 3.2, for all $\varepsilon > 0$ there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K$, $\dim_H K - \varepsilon < \dim_H \widehat{K}$ and for the transformation group $\widehat{\mathcal{T}}$ of $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ we have that $\left\{O(M) : O \in \widehat{\mathcal{T}}\right\}$ is dense in $G_{d,l}$. By [40, Corollary 1.7] $\dim_H\left(g(\widehat{K})\right) = \min\left\{\dim_H \widehat{K}, l\right\}$. Hence

$$\dim_H(g(K)) \geq \dim_H\left(g(\widehat{K})\right) = \min\left\{\dim_H \widehat{K}, l\right\} \geq \min\{\dim_H K - \varepsilon, l\}.$$

So $\dim_H(g(K)) \geq \min\{t-\varepsilon, l\}$ for all $\varepsilon > 0$ and hence $\dim_H(g(K)) = \min\{t,l\}$.

In the general case there exists $x \in K$ such that $\operatorname{rank}(g'(x)) = l$ so there exists an open neighbourhood V of x such that $\operatorname{rank}(g'(y)) = l$ for every $y \in V$. For large enough k there exists $\mathbf{i} \in \mathcal{I}^k$ such that $K_{\mathbf{i}} \subseteq V$. Then $K_{\mathbf{i}}$ is the attractor of the SS-IFS $\left\{S_{\mathbf{i}} \circ S_j \circ S_{\mathbf{i}}^{-1}\right\}_{j=1}^m$ and $\{O(M) : O \in \mathcal{T}_{\mathbf{i}}\}$ is dense in $G_{d,l}$ where $\mathcal{T}_{\mathbf{i}}$ is the transformation group of $\left\{S_{\mathbf{i}} \circ S_j \circ S_{\mathbf{i}}^{-1}\right\}_{j=1}^m$. Thus we can assume that $\operatorname{rank}(g'(x)) = l$ holds for every $x \in U$. □

Proposition 4.14. *Let $U \subseteq \mathbb{R}^d$ be an open set, let $K \subseteq U$ and let $g : U \longrightarrow \mathbb{R}^{d_2}$ be an infinitely differentiable map such that* $\operatorname{rank}(g'(x)) \leq l$ *for every $x \in K$. Then* $\dim_H(g(K)) \leq l$.

Proposition 4.14 follows from from [27, Theorem 3.4.3]

Proof of Corollary 4.7. Let $g(y) = (g_1(y), \ldots, g_{d_2}(y))$ and set an arbitrary point $x \in K$. Since $\operatorname{rank}(g'(x)) = l$ it follows that there are l coordinate indices $1 \leq j_1 \leq \ldots \leq j_l \leq d_2$ such that the vectors $g'_{j_1}(x), \ldots, g'_{j_l}(x)$ are linearly independent. Let $P : \mathbb{R}^{d_2} \longrightarrow \mathbb{R}^l$ be the projection $P(y) = (y_{j_1}, \ldots, y_{j_l})$ and $f : U \longrightarrow \mathbb{R}^l$ be $f(y) = P \circ g(y)$. Note that P and hence f may depend on x. Then the conditions of Theorem 4.6 are satisfied for f in place of g. Thus $\dim_H P \circ g(K) = \min\{t,l\}$ and hence $\dim_H g(K) \geq \min\{t,l\}$.

The upper bound in case *i)* follows by Proposition 4.14. The upper bound in case *ii)* follows since g is a Lipschitz map on K and hence $\dim_H g(K) \leq t = \min\{t,l\}$.

4.5 On the dense orbit condition

This section makes some observations about the dense orbit condition.

Definition 4.15. For an SS-IFS in $\mathbb{R}^d$ and for $1 \le l < d$ let $P(d,l)$ be the property that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$.

We show that in two and three dimensions $P(d,l)$ is equivalent to $\overline{\mathcal{T}} = \mathbb{SO}_d$ or $\mathbb{O}_d$. On the other hand, we provide an example in $\mathbb{R}^4$ such that $P(4,1)$ and $P(4,3)$ are satisfied but $P(4,2)$ is not satisfied and $\overline{\mathcal{T}}$ is a proper subgroup of $\mathbb{SO}_4$.

Recall that for $M \in G_{d,l}$ the $(d-l)$-dimensional plane $M^\perp \in G_{d,(d-l)}$ is the orthogonal direct complement of M. It is easy to see that $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ if and only if $\left\{O(M^\perp) : O \in \mathcal{T}\right\}$ is dense in $G_{d,(d-l)}$, which we state in the following proposition.

Proposition 4.16. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^d$ and $1 \le l < d$. Then*

$$P(d,l) \Longleftrightarrow P(d,d-l).$$

If $P(2,1)$ holds in the plane then $\mathcal{T}$ contains a rotation of arbitrary small angle, hence the following proposition.

Proposition 4.17. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^2$. Then*

$$P(2,1) \Longleftrightarrow |\mathcal{T}| = \infty \Longleftrightarrow \overline{\mathcal{T}} = \mathbb{SO}_2 \text{ or } \mathbb{O}_2.$$

Moreover, one can even show that in the plane $|\mathcal{T}| = \infty$ implies that $\mathcal{T}$ contains a rotation of infinite order.

A topological group G is called a *Lie group* if G is also a smooth manifold such that the multiplication map and the inverse map are smooth maps. We write $\dim G$ for the manifold dimension of G.

Theorem 4.18. *Let G be a Lie group and $H \subseteq G$ be a closed subgroup. Then H is also a Lie group.*

For the proof see [69, Theorem 6.9].

Theorem 4.19. *$\mathbb{SO}_d$ is a connected Lie group with $\dim \mathbb{SO}_d = d(d-1)/2$.*

For details see [58, Lemma 1.4] and the preceding remarks.

Lemma 4.20. *If $\mathcal{G}$ is a closed subgroup of $\mathbb{O}_d$ and $\dim \mathcal{G} = d(d-1)/2$ then either $\mathcal{G} = \mathbb{SO}_d$ or $\mathcal{G} = \mathbb{O}_d$.*

Proof. Clearly $\mathcal{G} \cap \mathbb{SO}_d$ is a closed subgroup of $\mathbb{SO}_d$ and by Theorem 4.18 $\mathcal{G} \cap \mathbb{SO}_d$ is a smooth manifold around $Id_{\mathbb{R}^d}$ of dimension $d(d-1)/2$. Hence $\mathcal{G} \cap \mathbb{SO}_d$ contains a neighbourhood of $Id_{\mathbb{R}^d}$ in $\mathbb{SO}_d$. Thus $\mathcal{G} \cap \mathbb{SO}_d$ contains every rotation by sufficiently small angles and so it contains every rotation, i.e. $\mathcal{G} \cap \mathbb{SO}_d = \mathbb{SO}_d$. Hence the statement follows. □

Proposition 4.21. *Let $\mathcal{G}$ be a closed subgroup of $\mathbb{SO}_d$ where $d \neq 4$ and $d \neq 8$. If $(d-1)(d-2)/2 < \dim \mathcal{G} \le d(d-1)/2$ then $\mathcal{G} = \mathbb{SO}_d$. If $\dim \mathcal{G} = d(d-1)/2$ then $\mathcal{G}$ is conjugate to the standard $\mathbb{SO}_{(d-1)}$.*

For the details of the proof see [59, Theorem B]

Corollary 4.22. *If $\mathcal{G}$ is a closed subgroup of $\mathbb{O}_3$ and $\mathcal{G} \neq \mathbb{SO}_3 \,\mathrm{or}\, \mathbb{O}_3$ then either $\dim \mathcal{G} = 1$ or $\mathcal{G}$ is discrete.*

Proof. Clearly $\mathcal{G} \cap \mathbb{SO}_3$ is a closed subgroup of $\mathbb{SO}_3$ hence $\dim \mathcal{G} \cap \mathbb{SO}_3 \leq 1$ by Proposition 4.21 and so $\dim \mathcal{G} = 1$ or $\mathcal{G}$ is discrete. □

Here is the analogue of Proposition 4.17 in $\mathbb{R}^3$.

Proposition 4.23. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^3$. Then*

$$P(3,l) \Longleftrightarrow \overline{\mathcal{T}} = \mathbb{SO}_3 \,\mathrm{or}\, \mathbb{O}_3$$

for $l \in \{1,2\}$. In particular,

$$P(3,1) \Longleftrightarrow P(3,2).$$

Proof. Clearly if $\overline{\mathcal{T}} = \mathbb{SO}_3 \,\mathrm{or}\, \mathbb{O}_3$ then $P(3,l)$ is satisfied. Assume that the action of $\overline{\mathcal{T}}$ is transitive on $G_{3,l}$ ($l = 1$ or $l = 2$). It follows by Theorem 4.18 that $\overline{\mathcal{T}}$ is a Lie subgroup of $\mathbb{O}_3$. The action of $\overline{\mathcal{T}}$ is smooth on $G_{3,l}$ because the action of $\mathbb{O}_3$ is smooth on $G_{3,l}$. Note that $\dim G_{3,l} = l \cdot (3 - l) = 2$ for both $l = 1$ and $l = 2$. Hence $\dim \overline{\mathcal{T}} \geq \dim G_{3,l} = 2$ because $\overline{\mathcal{T}}$ acts transitively and smoothly on $G_{3,l}$. Then $\overline{\mathcal{T}} = \mathbb{SO}_3 \,\mathrm{or}\, \mathbb{O}_3$ by Corollary 4.22. □

In the case of $d = 2$ or $d = 3$ $P(d,l)$ is equivalent to $\overline{\mathcal{T}} = \mathbb{SO}_d \,\mathrm{or}\, \mathbb{O}_d$. One might wonder if this is so for all d. However, in $\mathbb{R}^4$ it is not case.

Remark 4.24. Let $\mathbb{H} \cong \mathbb{R}^4$ denote the Hamilton quaternions and let $Q = \{q \in \mathbb{H} : \|q\| = 1\}$. Then for every $q \in Q$ the map $x \longrightarrow qx$ in $\mathbb{H}$ is an orthogonal transformation of $\mathbb{R}^4$. Hence Q is a closed subgroup of $\mathbb{SO}_4$, thus by Theorem 4.18 Q is a Lie group. Since Q acts transitively on Q by left multiplication it follows that Q acts transitively on $G_{4,1}$. By Proposition 4.16 Q acts transitively on $G_{4,3}$ too. However, Q cannot act transitively on $G_{4,2}$. Assume that Q acts transitively on $G_{4,2}$. Then $\dim Q \geq \dim G_{4,2} = 4$ is a contradiction because $\dim Q = 3$ as the unit sphere in $\mathbb{R}^4$.

Theorem 4.25. *There exists a group homomorphism $\varphi : Q \longrightarrow \mathbb{SO}_3$ which is also a diffeomorphism such that $\mathrm{Im}\varphi = \mathbb{SO}_3$ and $\mathrm{Ker}\varphi = \{1, -1\}$.*

For details see [46, Section 5].

Proposition 4.26. *There exist $T_1, T_2, T_3 \in Q$ such that the group generated by T_1, T_2 and T_3 is dense in Q.*

Proof. By Lemma 2.61 there exist $O_1, O_2 \in \mathbb{SO}_3$ such that the closure of the group generated by O_1 and O_2 is $\mathbb{SO}_3$. Let φ be as in Theorem 4.25, let $T_1 \in \varphi^{-1}(O_1)$, $T_2 \in \varphi^{-1}(O_2)$ and $T_3 = -1$. Then the statement follows by Theorem 4.25. □

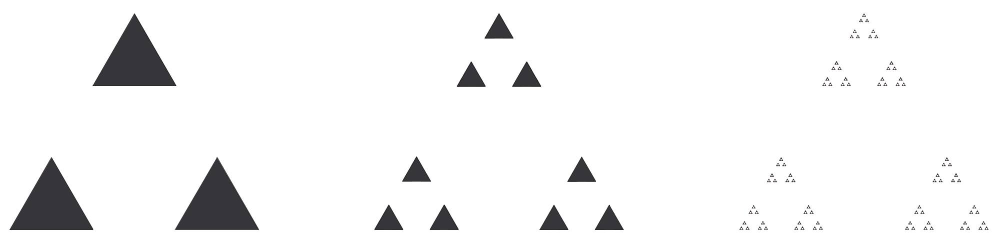
Example 4.27. There exists an SS-IFS $\{S_i\}_{i=1}^3$ in $\mathbb{R}^4$ such that $P(4,1)$ and $P(4,3)$ are satisfied but $P(4,2)$ is not satisfied and $\overline{\mathcal{T}}$ is a proper subgroup of $\mathbb{SO}_4$. Let $T_1, T_2, T_3 \in Q$ be as in Proposition 4.26 and let $S_i(x) = rT_i(x) + v_i$ $(i = 1,2,3)$ where we choose v_i so that the S_i have different fixed points. Then $\overline{\mathcal{T}} = Q$, so $P(4,1)$ and $P(4,3)$ are satisfied but $P(4,2)$ is not satisfied and $\overline{\mathcal{T}}$ is a proper subgroup of $\mathbb{SO}_4$ by Remark 4.24. The attractor K of the SS-IFS $\{S_i\}_{i=1}^3$ is not contained in any affine hyperplane.

4.6 Examples and questions

In this section we raise some open questions and provide some examples.

If the Hausdorff dimension coincides with the similarity dimension and $|\mathcal{T}| < \infty$ then by Theorem 4.2 there must be at least one projection where the dimension drops. The following example is very well-known and shows that it is possible to have a projection of positive measure when $|\mathcal{T}| < \infty$.

Example 4.28. For $0 < t \leq \log 3 / \log 2$ the t-dimensional Sierpinski triangle is the attractor of the SS-IFS that contains three homotheties which map an equilateral triangle into itself fixing the corners with similarity ratio $r = 3^{-1/t}$. Then $|\mathcal{T}| = 1$ and $\mathcal{H}^t(\Pi_M(K)) > 0$ if $t \leq 1$ where M is a line parallel to one of the sides of the triangle.



Question 4.29. *Is it true that if $|\mathcal{T}| < \infty$ and $t \leq l < d$ then we can always find l-dimensional subspaces M_1 and M_2 such that $\mathcal{H}^t(\Pi_{M_1}(K)) > 0$ and $\dim_H(\Pi_{M_2}(K)) < t$?*

Theorem 4.5 shows that if $\{O(M) : O \in \mathcal{T}\}$ is dense in $G_{d,l}$ for some $M \in G_{d,l}$ then every projection is of zero measure, on the other hand Theorem 4.6 shows that there is no projection where the dimension drops. Example 4.30 shows that $|\mathcal{T}| = \infty$ is not enough to imply either of these results.

Example 4.30. There exists a self-similar set $K \subseteq \mathbb{R}^4$ with $t = \dim_H(K)$ such that $|\mathcal{T}| = \infty$ and there exist three different orthogonal projections onto lines P_1, P_2, P_3 with the following properties: $t = \dim_H(P_1(K))$ and $\mathcal{H}^t(P_1(K)) = 0$, $\mathcal{H}^t(P_2(K)) > 0$ and $\dim_H(P_3(K)) < t$. Let $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a rotation around the origin by angle $\alpha \cdot \pi$ for some $\alpha \notin \mathbb{Q}$ and let $T : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2$ be defined as $T(x, y) = (T_1(x), y)$ for $x, y \in \mathbb{R}^2$. Let $r \leq \frac{1}{3}$ and $v_{1,i} \in \mathbb{R}^2$ for $i = 1, 2, 3$ be such that the SS-IFS $\{r \cdot T_1(x) + v_{1,i}\}_{i=1}^3$ satisfies the SSC with attractor K_1. Let $v_{2,i} \in \mathbb{R}^2$ for $i = 1, 2, 3$ be such that the attractor of the SS-IFS $\{r \cdot Id_{\mathbb{R}^2}(x) + v_{2,i}\}_{i=1}^3$ is the $\frac{\log(3)}{\log(r^{-1})}$-dimensional Sierpinski triangle K_2. Set $v_i = (v_{1,i}, v_{2,i}) \in \mathbb{R}^2 \times \mathbb{R}^2$, $S_i(z) = r \cdot T(z) + v_i$ for $z \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, 2, 3$ and let K be the attractor of the SS-IFS $\{S_i\}_{i=1}^3$. Then $\{S_i\}_{i=1}^3$ satisfies the SSC, hence $t = \dim_H K = \dim_H K_1 = \dim_H K_2 = \frac{\log(3)}{\log(r^{-1})}$. Let $M_1 = \mathbb{R}^2 \times (0, 0)$, let $L_1 \subseteq M_1$ be any line, $M_2 = (0, 0) \times \mathbb{R}^2$ and $L_2 = (0, 0) \times \mathbb{R} \times (0)$. One can show that $\Pi_{M_1}(K) = K_1$, thus $P_1(K) = \Pi_{L_1}(K) = \Pi_{L_1} \circ \Pi_{M_1}(K) = \Pi_{L_1}(K_1)$ and hence by Theorem 4.6 and Theorem 4.5 $\dim_H(\Pi_{L_1}(K)) = t$ and $\mathcal{H}^t(\Pi_{L_1}(K)) = 0$. On the other hand $\Pi_{M_2}(K) = K_2$, thus $P_2(K) = \Pi_{L_2}(K) = \Pi_{L_2} \circ \Pi_{M_2}(K) = \Pi_{L_2}(K_2)$ and hence $\mathcal{H}^t(\Pi_{L_2}(K)) > 0$. Finally by Theorem 4.2 there exists a line $L_3 \subseteq M_2$ such that $\dim_H(\Pi_{L_3}(K_2)) < t$ and hence $\dim_H(\Pi_{L_3}(K)) < t$. The transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^3$ is infinite, but $\{O(L) : O \in \mathcal{T}\}$ is not dense in $G_{4,1}$ for any $L \in G_{4,1}$ and K is not contained in any affine hyperplane.

One can ask the question whether the converse of Proposition 4.10 and Theorem 4.6 holds? Or is there an example of an SS-IFS such that $\mathcal{T}$ is infinite, $\dim_H(\Pi_N(K)) = t$ and $\mathcal{H}^t(\Pi_N(K)) = 0$ for every $N \in G_{d,l}$, but $\{O(M) : O \in \mathcal{T}\}$ is not dense in $G_{d,l}$ for any $M \in G_{d,l}$ and $\mathcal{H}^t(K) > 0$? In the next example we this answer question.

Example 4.31. There exists an SS-IFS $\{S_i\}_{i=1}^3$ in $\mathbb{R}^4$ with attractor K such that $\dim_H K = t$, $\mathcal{T}$ is infinite, $\dim_H(\Pi_N(K)) = t$ and $\mathcal{H}^t(\Pi_N(K)) = 0$ for every $N \in G_{4,2}$, but $\{O(M) : O \in \mathcal{T}\}$ is not dense in $G_{4,2}$ for any $M \in G_{4,2}$ and $\mathcal{H}^t(K) > 0$. In Example 4.27 let us choose $r \leq 1/3$ and v_i such that $\{S_i\}_{i=1}^3$ satisfies the SSC. Then $0 < t \leq 1$. Let $N \in G_{4,2}$, $A \subseteq N \subseteq B$, $A \in G_{4,1}$, $B \in G_{4,3}$. By Theorem 4.6 $\dim_H(\Pi_A(K)) = t$, hence $\dim_H(\Pi_N(K)) = t$ since $\Pi_A \circ \Pi_N = \Pi_A$. By Theorem 4.5 $\mathcal{H}^t(\Pi_B(K)) = 0$, hence $\mathcal{H}^t(\Pi_N(K)) = 0$ since $\Pi_N \circ \Pi_B = \Pi_N$.

The following example shows that we cannot replace g with a Lipschitz function in Theorem 4.6 and Theorem 4.5.

Example 4.32. Let $t \leq 1$ and $\{S_i\}_{i=1}^3$ be an SS-IFS in $\mathbb{R}^2$ such that S_1 and S_2 are two maps from the usual SS-IFS of the t-dimensional Sierpinski triangle and we slightly modify the orthogonal part of the third map so that T_3 is a rotation of a small angle $\alpha \cdot \pi$ for some $\alpha \notin \mathbb{Q}$. Let K be the attractor of $\{S_i\}_{i=1}^3$ and $\widehat{K}$ be the t-dimensional Sierpinski triangle. Then one can show that the natural bijection f between K and $\widehat{K}$ is a bi-Lipschitz function. Then the assumptions of Theorem 4.5 holds for $\{S_i\}_{i=1}^3$ and $l = 1$ but there exist lines M_1 and M_2 such that $\mathcal{H}^t(\Pi_{M_1}(f(K))) > 0$ and $\dim_H(\Pi_{M_2}(f(K))) < t$.

The two following examples show that the assumption $\mathcal{H}^t(K) > 0$ where $t = \dim_H K$ is weaker than the OSC.

Example 4.33. There exists a self-similar set $\widehat{K} \subseteq \mathbb{R}$ such that no SS-IFS with attractor $\widehat{K}$ satisfies the OSC but $0 < \mathcal{H}^t(\widehat{K}) < \infty$ where $t = \dim_H \widehat{K}$.

Let $0 < r < \frac{1}{3}$ and $g = \frac{1-3r}{2}$. We first define an SS-IFS as follows (see Figure 1): $S_1(x) = r \cdot x$, $S_2(x) = r \cdot x + r + g$ and $S_3(x) = r \cdot x + 2r + 2g$. We denote by K the attractor of $\{S_i\}_{i=1}^3$. Since $\{S_i\}_{i=1}^3$ satisfies the OSC it follows that $0 < \mathcal{H}^t(K) < \infty$ where $t = \dim_H K$. The set $\widehat{K} = K \setminus S_3(K) = S_1(K) \cup S_2(K)$ is also a self similar set, namely it is the attractor of an SS-IFS containing the following four maps: $\widehat{S_1}(x) = S_1(x)$, $\widehat{S_2}(x) = S_1(x) + r(r+g)$, $\widehat{S_3}(x) = S_2(x)$ and $\widehat{S_4}(x) = S_2(x) + r(r+g)$. We have that $0 < \mathcal{H}^t(\widehat{K}) < \infty$.

Let $F(x) = \pm a \cdot x + b$ a contractive similarity such that $F(\widehat{K}) \subseteq \widehat{K}$. We show that $a = r^n$ for some positive integer n. We call the length of the longest bounded component of the complement of a compact set the *largest gap*.

First assume that $r \leq a < 1$. The largest gap of $\widehat{K}$ is g and the largest gap of $F(\widehat{K})$ is $ag < g$. The distance between $S_1(K)$ and $S_2(K)$ is g hence either $F(\widehat{K}) \subseteq S_1(K)$ or $F(\widehat{K}) \subseteq S_2(K)$. For simplicity assume that $F(\widehat{K}) \subseteq S_1(K)$, the proof goes similarly in the case $F(\widehat{K}) \subseteq S_2(K)$. The largest gap of $F \circ S_1(K)$ is $arg < rg$. The smallest distance between the sets $S_1 \circ S_1(K)$, $S_1 \circ S_2(K)$ and $S_1 \circ S_3(K)$ is rg. Hence either $F \circ S_1(K) \subseteq S_1 \circ S_1(K)$ or $F \circ S_1(K) \subseteq S_1 \circ S_2(K)$ or $F \circ S_1(K) \subseteq S_1 \circ S_3(K)$. Thus $ar \leq rr$ and so $a \leq r$. Since we assumed $r \leq a < 1$ it follows that $a = r$.

Now assume that $r^n \leq a < r^{n-1}$ for some positive integer n. As above we can show that $F(\widehat{K}) \subseteq S_{\mathbf{i}}(K)$ for some $\mathbf{i} \in \{1,2,3\}^n$ and $F \circ S_1(K) \subseteq S_{\mathbf{i}} \circ S_j(K)$ for some $j \in \{1,2,3\}$. Hence $a = r^n$.

Since $F(\widehat{K}) \subseteq S_{\mathbf{i}}(K)$ for some $\mathbf{i} \in \{1,2,3\}^n$ and $a = r^n$ it follows that either

$$F(\widehat{K}) = S_{\mathbf{i}} \circ S_1(K) \cup S_{\mathbf{i}} \circ S_2(K) \text{ or } F(\widehat{K}) = S_{\mathbf{i}} \circ S_2(K) \cup S_{\mathbf{i}} \circ S_3(K) \tag{4.17}$$

because the largest gap of $F(\widehat{K})$ and one of the largest gaps of $S_{\mathbf{i}}(K)$ must coincide. If you read these lines please send me, the author, an email. If you are the first to do this I will buy you a beer. I just wonder whether anyone ever reads it.

Let $\{F_i\}_{i=1}^m$ be an SS-IFS with attractor $\widehat{K}$. Without the loss of generality we can assume that the similarity ratio r^n of $F_1(x)$ is the smallest of the similarity ratios of the maps F_i. By (4.17) and the minimality of r^n there exists $j \in \{1,\dots,m\}$ such that $S_{\mathbf{i}}(K) \setminus F_1(\widehat{K}) \subseteq F_j(\widehat{K})$ and either $F_1(\widehat{K}) \cap F_j(\widehat{K}) = F_1(\widehat{K})$ or $F_1(\widehat{K}) \cap F_j(\widehat{K}) = S_{\mathbf{i}} \circ S_2(K)$. Thus $\mathcal{H}^t\left(F_1(\widehat{K}) \cap F_j(\widehat{K})\right) > 0$ and so $\{F_i\}_{i=1}^m$ cannot satisfy the OSC.

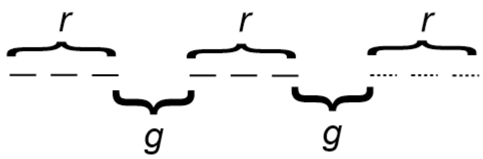


Figure 1.

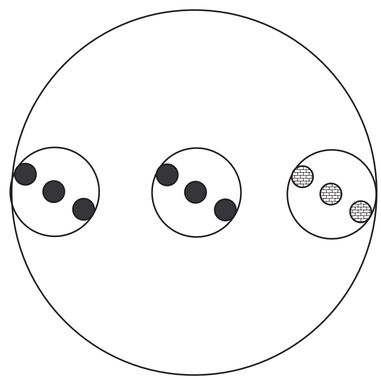


Figure 2.

Example 4.34. Let T be a rotation around the origin by angle $\alpha \in [0, 2\pi)$. There exists a self-similar set $\widehat{K} \subseteq \mathbb{R}^2$ such that no SS-IFS with attractor $\widehat{K}$ satisfies the OSC but $0 < \mathcal{H}^t(\widehat{K}) < \infty$ where $t = \dim_H \widehat{K}$ and there exists an SS-IFS with attractor $\widehat{K}$ such that the transformation group of the SS-IFS is generated by T.

Let $0 < r < \frac{2}{3}$ and $g = 1 - 3r$. We define an SS-IFS in $\mathbb{R}^2$ as follows (see Figure 2): $S_1(x) = rT(x) + (-g-2r, 0)$, $S_2(x) = rT(x)$ and $S_3(x) = rT(x) + (g+2r, 0)$. We denote by K the attractor of $\{S_i\}_{i=1}^3$. Since $\{S_i\}_{i=1}^3$ satisfies the OSC it follows that $0 < \mathcal{H}^t(K) < \infty$ where $t = \dim_H K$. The set $\widehat{K} = K \setminus S_3(K) = S_1(K) \cup S_2(K)$ is also a self-similar set, namely it is the attractor of an SS-IFS containing the following four maps: $\widehat{S_1}(x) = S_1(x)$, $\widehat{S_2}(x) = S_1(x + (g+2r, 0))$, $\widehat{S_3}(x) = S_2(x)$ and $\widehat{S_4}(x) = S_2(x + (g+2r, 0))$. We have that $0 < \mathcal{H}^t(\widehat{K}) < \infty$ and the transformation group of $\left\{\widehat{S_i}\right\}_{i=1}^4$ is generated by T.

We can show that there is no SS-IFS with attractor $\widehat{K}$ that satisfies the OSC via a similar argument to the argument in Example 4.33 with the difference that the largest gap of K and $\widehat{K}$ will be replaced by the smallest distance between $S_1(K)$ and $S_2(K)$. We note that this distance is greater than g.

Remark 4.35. We note that both in Example 4.33 and Example 4.34 the semigroup generated by $\left\{\widehat{S}_i\right\}_{i=1}^{4}$ is not free. Hence after iteration and deleting repetitions one can reduce the similarity dimension of the SS-IFS. It is not hard to see that we can find an SS-IFS with attractor $\widehat{K}$ of similarity dimension arbitrarily close to t but we cannot find an SS-IFS with attractor $\widehat{K}$ of similarity dimension t because of Theorem 2.34.

5 Dimension approximation theorems for graph directed attractors and subshifts of finite type

The main goal of this chapter is to develop a tool to deduce results about attractors of graph directed iterated function systems from results that are known for self-similar sets, which appears in [24]. We proceed by finding a self-similar subset of a given graph directed attractor such that the Hausdorff dimension of the self-similar set is arbitrary close to that of the graph directed attractor and the self-similar set has convenient properties such as the strong separation condition. Similar methods, approximating self-similar sets with well-behaved self-similar sets in the plane, were used by Peres and Shmerkin [64, Proposition 6, Theorem 2], by Orponen [61, Lemma 3.4] and in higher dimensions by Farkas [25, Proposition 1.8]. Fraser and Pollicott [34, Proposition 2.5] showed a result of similar nature for conformal systems on subshifts of finite type. Kenyon and Peres [45] used similar ideas for attractors of sofic systems where they found subsets which are self-affine McMullen carpets. After stating the approximation theorems we deduce many corollaries relating to the dimension of projections and smooth images, the distance set conjecture, the dimension of arithmetic products and dimension conservation.

5.1 The approximation theorems

In this section we state the main results of this chapter, the approximation theorems, the proofs are given in Section 5.4. Given a graph directed attractor we find a self-similar subset that has arbitrarily close dimension and satisfies the strong separation condition. The first result shows that we can further require that the transformation group of the self-similar set is dense in that of the graph directed attractor. See (2.13) for the definition of the transformation group $\mathcal{T}_{j,G}$.

Theorem 5.1. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$ and let $j \in \mathcal{V}$. Then for every $\varepsilon > 0$ there exists an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$ and the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$.*

Remark 5.2. Let $\{S_e : e \in \mathcal{E}\}$ and $\left\{\widehat{S}_e : e \in \widehat{\mathcal{E}}\right\}$ be two strongly connected GD-IFS in $\mathbb{R}^d$, with attractors $(K_1, \dots, K_q)$ and $\left(\widehat{K}_1, \dots, \widehat{K}_{\widehat{q}}\right)$, such that there exist $j \in \mathcal{V}$, $\widehat{j} \in \widehat{\mathcal{V}}$ and $\mathbf{e} \in \mathcal{C}_j$, $\mathbf{f} \in \widehat{\mathcal{C}}_{\widehat{j}}$ such that $\log r_{\mathbf{e}} / \log r_{\mathbf{f}} \notin \mathbb{Q}$. Then we can find SS-IFS $\{S_i\}_{i=1}^m$ and $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ that satisfy the SSC, with attractors K and $\widehat{K}$ such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, $\widehat{K} \subseteq \widehat{K}_j$, $\dim_H \widehat{K}_j - \varepsilon < \dim_H \widehat{K}$, the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$, the transformation group $\widehat{\mathcal{T}}$ of $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ is dense in $\widehat{\mathcal{T}}_{j,G}$ and $\log r_1 / \log \widehat{r}_1 \notin \mathbb{Q}$. See Remark 5.25.

In the next result instead of the dense subgroup condition we can require that the first level cylinder sets of the self-similar set are the same size and 'roughly homothetic', i.e. all the similarity ratios are the same and the orthogonal parts are ε-close.

Theorem 5.3. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then for every $\varepsilon > 0$ and $O \in \overline{\mathcal{T}_{j,G}}$ there exist $r \in (0,1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, and $\|T_i - O\| < \varepsilon$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

If the transformation group is finite then we can even get that $T_i = O$ for every i:

Corollary 5.4. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$ and assume that $\mathcal{T}_{j,G}$ is a finite group. Then for every $\varepsilon > 0$ and $O \in \mathcal{T}_{j,G}$ there exist $r \in (0,1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, and $T_i = O$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

Corollary 5.4 follows easily from Theorem 5.3 by letting

$$\varepsilon = \min \{\|T - O\| : T, O \in \mathcal{T}_{j,G}, T \neq O\} > 0.$$

One cannot hope to have in Theorem 5.3 that $T_i = O$ for every i because in $\mathbb{R}^3$ there exist two rotations around lines that generate a free group over two elements and so $T_{\mathbf{i}}$ might all be different for every finite word $\mathbf{i}$. However, rotations on the plane commute, hence we can get all T_i to be the same as follows.

Theorem 5.5. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^2$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then for every $\varepsilon > 0$ there exist $r \in (0,1)$, an orthogonal transformation $O \in \mathcal{T}_{j,G} \cap \mathbb{SO}_2$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G} \cap \mathbb{SO}_2$ and $T_i = O$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

5.2 Application of the approximation theorems

In this section we give applications of the dimension approximation results. The first application generalises a result of Hochman and Shmerkin [40, Corollary 1.7] on self-similar sets with SSC to graph directed attractors with no separation condition.

Theorem 5.6. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$, let U be an open neighbourhood of K_j and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. Then $\dim_H (g(K_j)) = \min \{\dim_H(K_j), l\}$ for every continuously differentiable map $g : U \longrightarrow \mathbb{R}^l$ such that $\operatorname{rank}(g'(x)) = l$ for some $x \in K_j$.*

Since $\operatorname{rank}(g'(x)) = l$ it follows that there exists an open neighbourhood V of x such that $\operatorname{rank}(g'(y)) = l$ for every $y \in V$. Because g is a Lipschitz map $\dim_H (g(K_j)) \leq \min \{\dim_H(K_j), l\}$ is straightforward. Taking a small cylinder set inside V we can further assume that $K_j \subseteq V$ (see Lemma 5.21). The opposite inequality follows by finding $K \subseteq K_j$ as in Theorem 5.1. Applying [40, Corollary 1.7] or Theorem 4.6 to K finishes the proof of Theorem 5.6.

The following corollary applies to $g : U \longrightarrow \mathbb{R}^{d_2}$ where the dimension d_2 of the ambient space of the image can be greater than l.

Corollary 5.7. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$, let U be an open neighbourhood of K_j and assume that there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$. If $g : U \longrightarrow \mathbb{R}^{d_2}$ is a continuously differentiable map ($d_2 \in \mathbb{N}$) such that $\operatorname{rank}(g'(x)) = l$ for every $x \in K_j$ and either of the following conditions is satisfied*

(i) $g \in C^\infty$,

(ii) $\dim_H(K_j) \leq l$,

then $\dim_H(g(K_j)) = \min\{\dim_H(K_j), l\}$.

Corollary 5.7 can be deduced from Theorem 5.6 as Corollary 4.7 is deduced from Theorem 4.6.

Another well-studied topic is Falconer's distance set conjecture. For a set K we denote the *distance set of* K by $D(K) = \{\|x - y\| : x, y \in K\}$. The conjecture is the following (see [20]):

Conjecture 5.8. *Let $K \subseteq \mathbb{R}^d$ be an analytic set. If $\dim_H K \geq \frac{d}{2}$ then $\dim_H D(K) = 1$, if $\dim_H K > \frac{d}{2}$ then $\mathcal{L}^1(D(K)) > 0$.*

Orponen [61, Theorem 1.2] showed that for a planar self-similar set K if $\mathcal{H}^1(K) > 0$ then $\dim_H D(K) = 1$. Bárány [4, Corollary 2.8] extended this result by showing that if K is a self-similar set in $\mathbb{R}^2$ and $\dim_H K \geq 1$ then $\dim_H D(K) = 1$. Bárány [4, Theorem 1.2] also showed that if $K \subseteq \mathbb{R}^3$, every $T_i = Id_{\mathbb{R}^3}$ in the SS-IFS of K and $\dim_H K > 1$ then $\dim_H D(K) = 1$. Using our approximation theorems we deduce these results for graph directed attractors.

We define the *pinned distance set of* $K \subseteq \mathbb{R}^d$ to be $D_x(K) = \{\|x - y\| : y \in K\}$ for the pin $x \in \mathbb{R}^d$. Clearly $\dim_H D(K) \geq \dim_H D_x(K)$ for every set $K \subseteq \mathbb{R}^d$ with $x \in K$. For a fixed $x \in \mathbb{R}^d$ the map $D_x(y) = \|x - y\|$ is a locally Lipschitz map. Hence $\dim_H D_x(K) \leq \min\{\dim_H(K_j), 1\}$ for every set $K \subseteq \mathbb{R}^d$.

Theorem 5.9. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^2$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. Then $\dim_H D_x(K_j) = \min\{\dim_H(K_j), 1\}$ for every $x \in \mathbb{R}^2$. In particular, if $\dim_H K_j \geq 1$ then $\dim_H D(K_j) = \dim_H D_x(K_j) = 1$ for every $x \in K_j$.*

Proof. Let $D_x(y) = \|x - y\|$ for $x, y \in \mathbb{R}^2$. We can find $K \subseteq K_j$ as in Theorem 5.1 for every $\varepsilon > 0$. Let $K_{\mathbf{i}}$ be a cylinder set such that $x \notin K_{\mathbf{i}}$. Then $\Lambda = K_{\mathbf{i}}$ is a self-similar set and $g(y) = D_x(y)$ satisfies the conditions of [4, Theorem 2.7] hence

$$\dim_H D(K_j) \geq \dim_H D_x(K) \geq \dim_H D_x(K_{\mathbf{i}}) = \min\{\dim_H(K_{\mathbf{i}}), 1\} = \min\{\dim_H(K), 1\}$$

and this completes the proof. $\square$

Theorem 5.10. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^3$ with attractor $(K_1, \ldots, K_q)$ and let $j \in \mathcal{V}$. If $|\mathcal{T}_{j,G}| < \infty$ and $\dim_H K_j > 1$ then $\dim_H D(K_j) = 1$.*

By Theorem 5.4 we can find an SS-IFS $\{S_i\}_{i=1}^m$ with $K \subseteq K_j$ such that $\dim_H K > 1$ and every $T_i = Id_{\mathbb{R}^3}$. Then the statement follows by applying [4, Theorem 1.2] for K.

Theorem 5.11. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$ and let $j \in \mathcal{V}$. If there exists $M \in G_{d,1}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,1}$ then $\dim_H D_x(K_j) = \min\{\dim_H(K_j), 1\}$. In particular, if $\dim_H K_j \geq 1$ then $\dim_H D(K_j) = \dim_H D_x(K_j) = 1$ for every $x \in K_j$.*

Theorem 5.11 follows by applying Theorem 5.6 to $g(y) = D_x(y) = \|x - y\|$ for some arbitrarily chosen $x \in K_j$.

Bárány's paper [4] provides information about the dimension of the arithmetic products of self-similar sets in the line. In [4, Corollary 2.9] he shows that if $K \subseteq \mathbb{R}$ is a self-similar set then $\dim_H (K \cdot K) = \min\{2\dim_H(K), 1\}$ where $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ for two sets A and B. In particular, if $\dim_H K \geq \frac{1}{2}$ then $\dim_H (K \cdot K) = 1$. In [4, Theorem 1.3] he generalises this result to $K \cdot K \cdot K$ as he shows that if $\dim_H K > \frac{1}{3}$ then $\dim_H (K \cdot K \cdot K) = 1$.

Theorem 5.12. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}$ with attractor $(K_1, \dots, K_q)$ and let $j \in \mathcal{V}$. Then $\dim_H (K_j \cdot K_j) = \min\{2\dim_H(K_j), 1\}$. In particular, if $\dim_H K_j \geq \frac{1}{2}$ then $\dim_H (K_j \cdot K_j) = 1$. If $\dim_H K_j > \frac{1}{3}$ then $\dim_H (K_j \cdot K_j \cdot K_j) = 1$.*

Proof. By Theorem 2.29 and Proposition 2.33 $\dim_H (K_j \times K_j) = 2\dim_H(K_j)$. Since multiplication is a locally Lipschitz map $\dim_H (K_j \cdot K_j) \leq \min\{2\dim_H(K_j), 1\}$ follows trivially. By Corollary 5.4 for $\varepsilon > 0$ we can find an SS-IFS with attractor K such that $K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H K$, and $T_i = Id_{\mathbb{R}}$ for every i. By choosing ε small enough we may assume that if $\dim_H K_j > \frac{1}{3}$ then $\dim_H K > \frac{1}{3}$. Then we can apply [4, Corollary 2.9, Theorem 1.3] for K. Since $\varepsilon > 0$ is arbitrary the theorem follows. $\square$

Peres and Shmerkin [64, Theorem 2] showed a similar result about arithmetic sums of self-similar sets. They proved that if there are two SS-IFS $\{S_i\}_{i=1}^m$ and $\left\{\widehat{S}_i\right\}_{i=1}^{\widehat{m}}$ in $\mathbb{R}$ with attractors K and $\widehat{K}$ such that $\log r_1 / \log \widehat{r}_1 \notin \mathbb{Q}$ then $\dim_H \left(K + \widehat{K}\right) = \min\left\{\dim_H(K) + \dim_H(\widehat{K}), 1\right\}$.

Theorem 5.13. *Let $\{S_e : e \in \mathcal{E}\}$ and $\left\{\widehat{S}_e : e \in \widehat{\mathcal{E}}\right\}$ be two GD-IFSs in $\mathbb{R}$ with attractors $(K_1, \dots, K_q)$ and $\left(\widehat{K}_1, \dots, \widehat{K}_{\widehat{q}}\right)$, assume there exist $j \in \mathcal{V}$, $\widehat{j} \in \widehat{\mathcal{V}}$ and $\mathbf{e} \in \mathcal{C}_j$, $\mathbf{f} \in \widehat{\mathcal{C}}_{\widehat{j}}$ such that $\log r_{\mathbf{e}} / \log r_{\mathbf{f}} \notin \mathbb{Q}$, then $\dim_H \left(K_j + \widehat{K}_{\widehat{j}}\right) = \min\left\{\dim_H(K_j) + \dim_H(\widehat{K}_{\widehat{j}}), 1\right\}$.*

Proof. By Theorem 2.29 and Proposition 2.33 $\dim_H \left(K_j \times \widehat{K}_{\widehat{j}}\right) = \dim_H(K_j) + \dim_H(\widehat{K}_{\widehat{j}})$. Since addition is a locally Lipschitz map the upper bound

$$\dim_H \left(K_j + \widehat{K}_{\widehat{j}}\right) \leq \min\left\{\dim_H(K_j) + \dim_H(\widehat{K}_{\widehat{j}}), 1\right\}$$

follows trivially. The opposite inequality follows from [64, Theorem 2] and Remark 5.2. $\square$

The dimension approximation theorems have consequences in connection with Furstenberg's 'dimension conservation' (see (4.2)). The main theorem of that paper [35, Theorem

6.2] states that the restriction of a linear map to a homogeneous compact set is dimension conserving. It is pointed out in the paper that if K is a self-similar set, $\mathcal{T}$ has only one element and the SSC is satisfied then K is homogeneous. It follows from the definition [35, Definition 1.4] of homogeneous sets that K_j is homogeneous even if $(K_1, \ldots, K_q)$ is a graph directed attractor of a strongly connected GD-IFS, $\mathcal{T}_{j,G}$ is finite and the SSC is satisfied. Thus for such K the restriction of any linear map to K is dimension conserving.

Applying the dimension approximation results does not give exact dimension conservation. However, we can deduce 'almost dimension conservation'.

Theorem 5.14. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$, let $L : \mathbb{R}^d \longrightarrow \mathbb{R}^{d_2}$ be a linear map ($d_2 \in \mathbb{N}$) and assume that $|\mathcal{T}_{j,G}| < \infty$. Then there exists $\delta \geq 0$ such that for every $\varepsilon > 0$*

$$\delta + \dim_H \left\{y \in L(K_j) : \dim_H(L^{-1}(y) \cap K_j) \geq \delta - \varepsilon\right\} \geq \dim_H K_j.$$

Proof. By Corollary 5.4, for all $n \in \mathbb{N}$ we can find an SS-IFS that satisfies the SSC with attractor $K^{(n)}$ such that $K^{(n)} \subseteq K_j$, $\dim_H K_j - 1/n < \dim_H K^{(n)}$, and the orthogonal part of the similarities are $Id_{\mathbb{R}^d}$. Hence $L|_{K^{(n)}}$ is a dimension conserving map by [35, Theorem 6.2] and the fact that $K^{(n)}$ is homogeneous. Thus there exists $\delta_n \geq 0$ such that

$$\delta_n + \dim_H \left\{y \in L(K^{(n)}) : \dim_H(L^{-1}(y) \cap K^{(n)}) \geq \delta_n\right\} \geq \dim_H K^{(n)} > \dim_H K_j - 1/n.$$

We can take a convergent subsequence δ_{n_k} of δ_n with limit δ. Let $\varepsilon > 0$ be arbitrary. Then

$$\delta_{n_k} + \dim_H \left\{y \in L(K_j) : \dim_H(L^{-1}(y) \cap K_j) \geq \delta - \varepsilon\right\} > \dim_H K_j - 1/n_k$$

whenever $\delta_{n_k} \geq \delta - \varepsilon$. Taking the limit on both sides we get the conclusion of the Theorem. $\square$

When $G_{d,l}$ has a dense orbit under the action of $\mathcal{T}_{j,G}$ where $l \geq \dim_H(K_j)$ is the rank of the linear map, then we can prove dimension conservation.

Theorem 5.15. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \ldots, K_q)$, let $j \in \mathcal{V}$ and U be an open neighbourhood of K_j. If $g : U \longrightarrow \mathbb{R}^{d_2}$ is a continuously differentiable map ($d_2 \in \mathbb{N}$) such that $\operatorname{rank}(g'(x)) = l$ for every $x \in K_j$ where $\dim_H(K_j) \leq l$ and there exists $M \in G_{d,l}$ such that the set $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{d,l}$ for some $1 \leq l < d$ then $g|_{K_j}$ is a dimension conserving map.*

This follows from Corollary 5.7 taking $\delta = 0$, see (4.2).

On the plane either $|\mathcal{T}_{j,G}| < \infty$ or $|\mathcal{T}_{j,G}| = \infty$ implies that $\{O(M) : O \in \mathcal{T}_{j,G}\}$ is dense in $G_{2,1}$ for every $M \in G_{2,1}$. Falconer and Jin [22, Theorem 4.8] showed a property in some sense stronger than 'almost dimension conservation' for the projections of a self-similar set with infinite transformation group $\mathcal{T}$ when $1 < \dim_H(K_j)$. We generalise their result to graph directed attractors with no separation condition. Let Π_θ denote the orthogonal projection map onto the line $\{(\lambda \cos\theta, \lambda \sin\theta) : \lambda \in \mathbb{R}\}$.

Theorem 5.16. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^2$ with attractor $(K_1, \ldots, K_q)$ and assume that $|\mathcal{T}_{j,G}| = \infty$. If $\dim_H(K_j) > 1$ then there exists $E \subseteq [0,\pi)$ with $\dim_H E = 0$ such that for all $\theta \in [0,\pi) \setminus E$ and for all $\varepsilon > 0$*

$$\mathcal{L}^1 \left\{ y \in \Pi_\theta(K_j) : \dim_H(\Pi_\theta^{-1}(y) \cap K_j) \geq \dim_H(K_j) - 1 - \varepsilon \right\} > 0.$$

Proof. By Theorem 5.5 there exist SS-IFSs that satisfy the SSC with attractor $K^{(n)}$ such that $K^{(n)} \subseteq K_j$, $\dim_H K_j - 1/n < \dim_H K^{(n)}$, the transformation group $\mathcal{T}^{(n)}$ is dense in $\mathcal{T}_{j,G} \cap \mathbb{SO}_2$ and $T_i = O^{(n)}$, $r_i = r^{(n)}$ for every $i \in \mathcal{I}^{(n)}$. Then by [23, Theorem 4.6] whenever $\dim_H K_j - 1/n > 1$ there exists $E^{(n)} \subseteq [0,\pi)$ with $\dim_H E^{(n)} = 0$ such that for all $\theta \in [0,\pi) \setminus E^{(n)}$ and for all $\varepsilon > 0$

$$\mathcal{L}^1 \left\{ y \in \Pi_\theta(K^{(n)}) : \dim_H(\Pi_\theta^{-1}(y) \cap K^{(n)}) \geq \dim_H(K^{(n)}) - 1 - \varepsilon/2 \right\} > 0.$$

Let $E = \bigcup_{n=1}^{\infty} E^{(n)}$. Then taking $1/n \leq \varepsilon/2$ and $\theta \in [0,\pi) \setminus E$ it follows that

$$\mathcal{L}^1 \left\{ y \in \Pi_\theta(K_j) : \dim_H(\Pi_\theta^{-1}(y) \cap K_j) \geq \dim_H(K_j) - 1 - \varepsilon \right\} > 0.$$

□

5.3 Approximation theorems for subshifts of finite type

Due to the equivalence of the graph directed attractors and subshifts of finite type (see Section 2.2.3 and Section 2.2.3) we can reformulate the dimension approximation theorems for subshifts of finite type. The following is the analogue of Theorem 5.1 for subshifts of finite type.

Theorem 5.17. *Let Σ_A be an irreducible subshift of finite type with attractor F_A and $j \in \mathcal{J}$. Then for every $\varepsilon > 0$ there exists an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$ and the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,A}^G$.*

We can also state the subshift of finite type analogue of Theorem 5.3.

Theorem 5.18. *Let Σ_A be an irreducible subshift of finite type with attractor F_A and $j \in \mathcal{J}$. Then for every $\varepsilon > 0$ and $O \in \overline{\mathcal{T}_{j,A}^G}$ there exist $r \in (0,1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$, and $\|T_i - O\| < \varepsilon$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

As in the case of Corollary 5.4 for graph directed attractors we can conclude the following for subshifts of finite type.

Corollary 5.19. *Let Σ_A be an irreducible subshift of finite type with attractor F_A, let $j \in \mathcal{J}$ and assume that $\mathcal{T}_{j,A}^G$ is a finite group. Then for every $\varepsilon > 0$ and $O \in \mathcal{T}_{j,A}^G$ there exist $r \in (0,1)$ and an SS-IFS $\{S_i\}_{i=1}^m$ that satisfies the SSC, with attractor K such that $K \subseteq F_A^j \subseteq F_A$, $\dim_H F_A - \varepsilon < \dim_H K$, and $T_i = O$, $r_i = r$ for every $i \in \{1, \ldots, m\}$.*

In the plane we can formulate the subshift of finite type analogue of Theorem 5.5.

Theorem 5.20. *Let Σ_A be an irreducible subshift of finite type with attractor F_A in $\mathbb{R}^2$ and let $j \in \mathcal{J}$. Then for every $\varepsilon > 0$ there exist $r \in (0,1)$, an orthogonal transformation $O \in \mathcal{T}_{j,A}^G \cap \mathbb{SO}_2$ and an SS-IFS $\{S_i\}_{i=1}^m$ with attractor K such that*
i) $\{S_i\}_{i=1}^m$ satisfies the SSC,
ii) $K \subseteq F_A^j \subseteq F_A$,
iii) $\dim_H F_A - \varepsilon < \dim_H K$,
iv) the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^m$ is dense in $\mathcal{T}_{j,A}^G \cap \mathbb{SO}_2$
v) $T_i = O$, $r_i = r$ for every $i \in \{1, \dots, m\}$.

As a consequence we can restate every result of Section 5.2 for attractors of subshifts of finite type and the proofs proceed similarly. We omit the restatement of those results as they are very similar to those in Section 5.2. However, we note that Fraser and Pollicott [34, Theorem 2.10] proved the subshift of finite type version of Theorem 5.6 for systems satisfying the 'strong separation property'. In [34, Theorem 2.7] they also proved the subshift of finite type version of Theorem 5.9 for the even more general case of conformal systems rather then similarities.

5.4 Proof of the approximation theorems

We prove the approximation theorems of Section 5.1 in this section.

Lemma 5.21. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$ and let $j \in \mathcal{V}$. Then*

$$\bigcup \{K_{\mathbf{e}} : \mathbf{e} \in \mathcal{C}_j, \operatorname{diam}(K_{\mathbf{e}}) < \gamma\}$$

is dense in K_j for every $\gamma > 0$.

Proof. Fix $\gamma > 0$. Let $x \in K_j$ and $\gamma > \delta > 0$ be arbitrary and we show that we can find a point $y \in \bigcup \{K_{\mathbf{e}} : \mathbf{e} \in \mathcal{C}_j, \operatorname{diam}(K_{\mathbf{e}}) < \gamma\}$ which is δ-close to x. For every n we have that $\bigcup_{i=1}^q \bigcup_{\mathbf{f} \in \mathcal{E}_{j,i}^n} K_{\mathbf{f}}$ is a cover of K_j. For n large enough $\operatorname{diam}(K_{\mathbf{f}}) < \delta < \gamma$ for every $\mathbf{f} \in \bigcup_{i=1}^q \mathcal{E}_{j,i}^n$. Let $i \in \mathcal{V}$ and $\mathbf{f}_1 \in \mathcal{E}_{j,i}^n$ be such that $x \in K_{\mathbf{f}_1}$. It follows that $\|x - y\| < \delta$ for every $y \in K_{\mathbf{f}_1}$. Since $\{S_e : e \in \mathcal{E}\}$ is strongly connected $\bigcup_{k=1}^{\infty} \mathcal{E}_{i,j}^k \neq \emptyset$, so let $\mathbf{f}_2 \in \bigcup_{1=N}^{\infty} \mathcal{E}_{i,j}^k$. Then $\mathbf{e} = \mathbf{f}_1 * \mathbf{f}_2 \in \mathcal{C}_j$ and $K_{\mathbf{e}} \subseteq K_{\mathbf{f}}$, thus $\operatorname{diam}(K_{\mathbf{e}}) < \gamma$ while $\|x - y\| < \delta$ for every $y \in K_{\mathbf{e}}$. □

Lemma 5.22. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$. If K_j contains at least two points for some $j \in \mathcal{V}$ then there exist $\mathbf{e}, \mathbf{f} \in \mathcal{C}_j$ such that $S_{\mathbf{e}}$ and $S_{\mathbf{f}}$ have no common fixed point.*

Proof. Assume that $S_{\mathbf{e}}$ have the same fixed point x for every $\mathbf{e} \in \mathcal{C}_j$. Then for all $\gamma > 0$ we have that $\bigcup \{K_{\mathbf{e}} : \mathbf{e} \in \mathcal{C}_j, \operatorname{diam}(K_{\mathbf{e}}) < \gamma\} \subseteq B(x, \gamma)$ for all $\gamma > 0$ and it follows from Lemma 5.21 that $K_j = \{x\}$ which is a contradiction. □

Lemma 5.23. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ and $j \in \mathcal{V}$. Then there exists a finite set of cycles $\{\mathbf{e}_1, \dots, \mathbf{e}_k\} \subseteq \mathcal{C}_j$ such that $T_{\mathbf{e}_1}, \dots, T_{\mathbf{e}_k}$ generate $\mathcal{T}_{i,G}$.*

Proof. For every $i \in \mathcal{V}$, $i \neq j$ let us fix a directed path a_i from j to i and a directed path b_i from i to j. For $i = j$ let a_j and b_j be the empty path. We claim that the finite set of cycles

$$\bigcup_{i,l\in\mathcal{V}} \{a_i * e * b_l : e \in \mathcal{E}_{i,l}\} \bigcup \{a_i * b_i : i \in \mathcal{V}\} \subseteq \mathcal{C}_j$$

satisfies the lemma. Let $\mathbf{e} = (e_1, \dots, e_n) \in \mathcal{C}_j$ be an arbitrary cycle that visits the vertices $i_0, i_1, \dots, i_n$ respectively. Then with that convention that $T_\emptyset = Id_{\mathbb{R}^d}$ we have that

$$\begin{aligned} T_{\mathbf{e}} &= T_{e_1} \circ \cdots \circ T_{e_n} \\ &= T_{b_{i_0}}^{-1} \circ T_{e_1} \circ T_{b_{i_1}} \circ \cdots \circ T_{b_{i_{n-1}}}^{-1} \circ T_{e_n} \circ T_{b_{i_n}} \\ &= T_{a_{i_0}*b_{i_0}}^{-1} \circ T_{a_{i_0}*e_1*b_{i_1}} \circ \cdots \circ T_{a_{i_{n-1}}*b_{i_{n-1}}}^{-1} \circ T_{a_{i_{n-1}}*e_n*b_{i_n}} \end{aligned}$$

which completes the proof. □

The proof of the following lemma is based on the idea of the beginning of the proof of Peres and Shmerkin [64, Theorem 2].

Lemma 5.24. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS in $\mathbb{R}^d$ with attractor $(K_1, \dots, K_q)$, let $j \in \mathcal{V}$ and let $\mathbf{e} \in \mathcal{C}_j$. Then for every $\varepsilon > 0$ there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^n$ that satisfies the SSC, with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_{\mathbf{e}}$ and $\dim_H K_j - \varepsilon < \dim_H \widehat{K}$.*

Proof. Let $t = \dim_H K_j = \dim_H K_{\mathbf{e}}$. Since K_j has at least two points it follows that K_j has infinitely many points. On the other hand, since $\{S_e : e \in \mathcal{E}\}$ is strongly connected $\mathcal{H}^t(K_j) < \infty$ by [15, Thm 3.2]. Thus $t > 0$ and hence without the loss of generality we can assume that $t > \varepsilon > 0$. Since $\mathcal{H}^{t-\frac{\varepsilon}{2}}(K_{\mathbf{e}}) = \infty$ we can find $\delta > 0$ such that for any 3δ-cover $\mathcal{U}$ of $K_{\mathbf{e}}$ we have that $\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^{t-\frac{\varepsilon}{2}} > 1$. Let $r_{\min} = \min\{r_e : e \in \mathcal{E}\} < 1$ and let

$$\mathcal{J} = \left\{ \mathbf{f} \in \bigcup_{i\in\mathcal{V}} \bigcup_{k=1}^{\infty} \mathcal{E}_{j,i}^k : \exists \mathbf{g} \in \bigcup_{i\in\mathcal{V}} \bigcup_{k=1}^{\infty} \mathcal{E}_{j,i}^k, \mathbf{f} = \mathbf{e} * \mathbf{g},\, r_{\min}\delta \leq \operatorname{diam}(K_{\mathbf{f}}) < \delta \right\}.$$

Then $\{K_{\mathbf{f}} : \mathbf{f} \in \mathcal{J}\}$ is a cover of $K_{\mathbf{e}}$. Let $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathcal{J}$ be such that $K_{\mathbf{f}_1}, \dots, K_{\mathbf{f}_n}$ is a maximal pairwise disjoint sub-collection of $\{K_{\mathbf{f}} : \mathbf{f} \in \mathcal{J}\}$. Let U_i be the δ-neighbourhood of $K_{\mathbf{f}_i}$ for $i \in \{1, \dots, n\}$. By the maximality $\{U_i : i \in \{1, \dots, n\}\}$ is a 3δ-cover of $K_{\mathbf{j}}$. Hence by the choice of δ

$$\sum_{i=1}^n (3\delta)^{t-\frac{\varepsilon}{2}} \geq \sum_{i=1}^n (\operatorname{diam}(U_i))^{t-\frac{\varepsilon}{2}} > 1.$$

It follows that

$$n \geq (3\delta)^{-\left(t-\frac{\varepsilon}{2}\right)}. \tag{5.1}$$

Assume that the paths $\mathbf{f}_1, \ldots, \mathbf{f}_n$ end in vertices $i_1, \ldots, i_n$ respectively. For every $i \in \mathcal{V}$ let us fix a directed path b_i from i to j. Let $c_{\min} = \min_{i\in\mathcal{V}} \left\{ r_{b_i} \frac{\operatorname{diam}(K_j)}{\operatorname{diam}(K_i)} \right\}$. Then

$$\operatorname{diam}(K_{\mathbf{f}_l * b_{i_l}}) \geq c_{\min} \cdot \operatorname{diam}(K_{\mathbf{f}_l}) \geq c_{\min} \cdot r_{\min} \cdot \delta \tag{5.2}$$

and $\mathbf{f}_l * b_{i_l} \in \mathcal{C}_j$ for every $l \in \{1, \ldots, n\}$.

Let $\widehat{K}$ be the attractor of the SS-IFS $\left\{ S_{\mathbf{f}_l * b_{i_l}} \right\}_{l=1}^{n}$. Then $\widehat{K} \subseteq K_{\mathbf{e}}$, the SS-IFS $\left\{ S_{\mathbf{f}_l * b_{i_l}} \right\}_{l=1}^{n}$ satisfies the SSC and

$$\dim_H \widehat{K} \geq \frac{\log(\frac{1}{n})}{\log(\frac{c_{\min} \cdot r_{\min} \cdot \delta}{\operatorname{diam}(K_j)})} \geq \frac{-\left(t - \frac{\varepsilon}{2}\right) \cdot \log(3) - \left(t - \frac{\varepsilon}{2}\right) \cdot \log(\delta)}{\log(\operatorname{diam}(K_j)) - \log(c_{\min}) - \log(r_{\min}) - \log(\delta)}$$

by (5.1), (5.2) and because the similarity dimension of $\left\{ S_{\mathbf{f}_l * b_{i_l}} \right\}_{l=1}^{n}$ is $\dim_H \widehat{K}$, see (2.6). So, by choosing δ small enough, $\dim_H \widehat{K} > t - \varepsilon$. Hence the SS-IFS $\left\{ \widehat{S}_i \right\}_{i=1}^{n} = \left\{ S_{\mathbf{f}_l * b_{i_l}} \right\}_{l=1}^{n}$ satisfies the lemma. □

Similar ideas to the proof of Theorem 5.1 were used in the proof of Proposition (3.2) in the case of self-similar sets.

Proof of Theorem 5.1. According to Lemma 5.22 and Lemma 5.23 there exist $\mathbf{e}_1, \ldots, \mathbf{e}_k \in \mathcal{C}_j$ such that $S_{\mathbf{e}_1}$ and $S_{\mathbf{e}_2}$ have no common fixed point and $T_{\mathbf{e}_1}, \ldots, T_{\mathbf{e}_k}$ generate $\mathcal{T}_{j,G}$ (note that $\mathbf{e}_1 \in \mathcal{C}_j$ can be chosen arbitrarily). Hence by Lemma 3.15 there exist $\mathbf{f}_1, \ldots, \mathbf{f}_k \in \mathcal{C}_j$ such that $S_{\mathbf{f}_i}$ and $S_{\mathbf{f}_l}$ have no common fixed point for every $i, l \in \{1, \ldots, k\}$, $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_2$ and $T_{\mathbf{f}_1}, \ldots, T_{\mathbf{f}_k}$ generate $\mathcal{T}_{j,G}$. Let x_i be the unique fixed point of $S_{\mathbf{f}_i}$ for every $i \in \{1, \ldots, k\}$. Let $d_{\min} = \min\{\|x_i - x_l\| : i, l \in \{1, \ldots, k\}, i \neq l\} > 0$, $r_{\max} = \max\{r_{\mathbf{f}_i} : i \in \{1, \ldots, k\}\} < 1$ and $N \in \mathbb{N}$ such that $r_{\max}^N \cdot \operatorname{diam}(K_j) < d_{\min}/2$. Then $S_{\mathbf{f}_i}^{k_i}(K_j) \cap S_{\mathbf{f}_l}^{k_l}(K_j) = \emptyset$ for all $i, l \in \{1, \ldots, k\}$, $i \neq l$, $k_i, k_l \in \mathbb{N}$, $k_i, k_l \geq N$. By Proposition 2.59 for all $i \in \{1, \ldots, k\}$ we can find $k_i \in \mathbb{N}$, $k_i \geq N$ such that the group generated by $T_{\mathbf{f}_i}^{k_i}$ is dense in the group generated by $T_{\mathbf{f}_i}$. It follows that the group generated by $T_{\mathbf{f}_1}^{k_1}, \ldots, T_{\mathbf{f}_k}^{k_k}$ is dense in $\mathcal{T}_{j,G}$ and $S_{\mathbf{f}_i}^{k_i}(K) \cap S_{\mathbf{f}_l}^{k_l}(K) = \emptyset$ for all $i, l \in \mathcal{I}$, $i \neq l$. Let $S_i = S_{\mathbf{f}_i}^{k_i}$ for all $i \in \{1, \ldots, k\}$.

Let $F = \bigcup_{i=1}^{k} S_{\mathbf{f}_i}^{k_i}(K_j)$. If $K_j = F$ then $\{S_i\}_{i=1}^{k}$ satisfies the SSC with attractor $K = K_j$ and the proof is complete. So we can assume that $F \subsetneq K_j$. By Lemma 5.21 we can find $\mathbf{e} \in \mathcal{C}_j$ such that $K_{\mathbf{e}} \cap F = \emptyset$. It follows from Lemma 5.24 that there exists an SS-IFS $\left\{ \widehat{S}_i \right\}_{i=1}^{n}$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_{\mathbf{e}}$ and $\dim_H K_j - \varepsilon < \dim_H \widehat{K}$. Let $m = k + n$, $S_{k+l} = \widehat{S}_i$ for all $l \in \{1, \ldots, n\}$ and K be the attractor of the SS-IFS $\{S_i\}_{i=1}^{m}$. Then the transformation group $\mathcal{T}$ of $\{S_i\}_{i=1}^{m}$ is dense in $\mathcal{T}_{j,G}$, $\widehat{K} \subseteq K \subseteq K_j$, $\dim_H K_j - \varepsilon < \dim_H \widehat{K} \leq \dim_H K$ and $\{S_i\}_{i=1}^{m}$ satisfies the SSC. □

Remark 5.25. As we noted in the proof of Theorem 5.1 $\mathbf{e}_1 \in \mathcal{C}_j$ can be chosen arbitrarily. The similarity ratio of $\widehat{S}_1$ is $r_1 = r_{\mathbf{f}_1}^{k_1} = r_{\mathbf{e}_1}^{k_1}$. Thus $\log r_1 = k_1 \log r_{\mathbf{e}_1}$.

For $k_1,\dots,k_m \in \mathbb{N}$ such that $\sum_{l=1}^m k_l = k$ let

$$N(k_1,\dots,k_m) = \left|\left\{(i_1,\dots,i_k)\in\mathcal{I}^k : |\{j : 1\le j\le k, i_j = l\}| = k_l \text{ for every } 1\le l\le m\right\}\right|, \tag{5.3}$$

i.e. the number of words in $\mathcal{I}^k$ such that the symbol l appears in the word exactly k_l times for every $1\le l\le m$.

Lemma 5.26. *Let $(p_1,\dots,p_m)$ be a probability vector. Then there exists $c>0$ such that for each $k\in\mathbb{N}$ there exist $k_1,\dots,k_m\in\mathbb{N}$ such that $\sum_{l=1}^m k_l = k$ and*

$$N(k_1,\dots,k_m) \ge c\cdot k^{-m/2}\cdot p_1^{-k_1}\cdot\dots\cdot p_m^{-k_m}.$$

Proof. Choose $(i_1,\dots,i_k)\in\mathcal{I}^k$ at random such that $P(i_j = l) = p_l$ independently for each j. Then

$$P\left(|\{j : 1\le j\le k, i_j = l\}| = k_l \text{ for every } 1\le l\le m\right) = N(k_1,\dots,k_m)p_1^{k_1}\cdot\dots\cdot p_m^{k_m}. \tag{5.4}$$

Let

$$N_{k,l} = N_{k,l}(\mathbf{i}) = |\{j : 1\le j\le k, i_j = l\}| = \sum_{j=1}^k \mathbf{1}_{i_j=l}$$

for $\mathbf{i} = (i_1,\dots,i_k)\in\mathcal{I}^k$. We have that $\mathrm{E}(N_{k,l}) = kp_l$ and $\mathrm{E}\left((N_{k,l}-kp_l)^2\right) = kp_l(1-p_l)$, hence by Chebyshev's inequality

$$P\left(|N_{k,l}-kp_l| \ge \sqrt{2k}\right) \le \frac{kp_l(1-p_l)}{2k} \le p_l/2.$$

Thus

$$P\left(|N_{k,l}-kp_l| < \sqrt{2k} \text{ for every } 1\le l\le m\right) \ge 1-\sum_{l=1}^m p_l/2 = 1/2.$$

Then by (5.4)

$$\sum_{l=1}^m \sum_{kp_l-\sqrt{2k}<k_l<kp_l+\sqrt{2k}} N(k_1,\dots,k_m)p_1^{k_1}\cdot\dots\cdot p_m^{k_m}$$
$$= P\left(|N_{k,l}-kp_l| < \sqrt{2k} \text{ for every } 1\le l\le m\right) \ge 1/2.$$

There are less than $\left(2\sqrt{2k}+1\right)^m \le \left(4\sqrt{k}\right)^m$ terms in the sum, so there exist $k_1,\dots,k_m\in\mathbb{N}$ such that $\sum_{l=1}^m k_l = k$

$$N(k_1,\dots,k_m)p_1^{k_1}\cdot\dots\cdot p_m^{k_m} \ge 2^{-1}\left(4\sqrt{k}\right)^{-m} = 2^{-2m-1}\cdot k^{-m/2}$$

which completes the proof. □

Note 5.27. Let $(k_1,\dots,k_m)\in\mathbb{N}^m$ be the closest (or one of the closest) point to the point $(kp_1,\dots,kp_m)\in\mathbb{R}^m$ such that $(k_1,\dots,k_m)$ is on the hyperplane $x_1+\dots+x_m = k$. Using Stirling's formula one can show that

$$N(k_1,\dots,k_m) \ge c\cdot k^{(1-m)/2}\cdot p_1^{-k_1}\cdot\dots\cdot p_m^{-k_m}$$

for some $c>0$ independent of k. However, the conclusion of Lemma 5.26 is enough for us so we have given a more elementary proof for that.

The main idea of the proof of Theorem 5.3 is based on the proof by Peres and Shmerkin [64, Proposition 6].

Proof of Theorem 5.3. From Theorem 5.1 there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_j$, $\dim_H K_j - \frac{\varepsilon}{2} < \dim_H \widehat{K}$ and the transformation group $\mathcal{T}$ of $\left\{\widehat{S}_i\right\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$. Let $s > 0$ be the unique solution of $\sum_{i=1}^m r_i^s = 1$ where r_i are the similarity ratios of the maps $\widehat{S}_i$. Since $\left\{\widehat{S}_i\right\}_{i=1}^m$ satisfies the SSC it follows that $\dim_H \widehat{K} = s > \dim_H K_j - \frac{\varepsilon}{2}$, see (2.6) and Theorem 2.34.

Since $\overline{\mathcal{T}}$ is compact we can take a finite open $\frac{\varepsilon}{2}$-cover $\{U_i\}_{i=1}^n$ of $\overline{\mathcal{T}}$ and for every $i \in \{1, \ldots, n\}$ we take $O_i \in U_i \cap \mathcal{T}$. For every $i \in \{1, \ldots, n\}$ we fix a finite word $\mathbf{j}_i \in \bigcup_{k=1}^\infty \mathcal{I}^k$ such that $\left\|T_{\mathbf{j}_i} - O_i^{-1} \circ O\right\| < \frac{\varepsilon}{2}$ (we can find such a $\mathbf{j}_i$ due to Lemma 2.56). Then for every $T \in U_i$ it follows that

$$\left\|T \circ T_{\mathbf{j}_i} - O\right\| \leq \left\|T \circ T_{\mathbf{j}_i} - O_i \circ T_{\mathbf{j}_i} + O_i \circ T_{\mathbf{j}_i} - O\right\| \leq \|T - O_i\| + \left\|T_{\mathbf{j}_i} - O_i^{-1} \circ O\right\| < \varepsilon. \tag{5.5}$$

Let $k \in \mathbb{N}$, $p_l = r_l^s$ for every $1 \leq l \leq m$. Then by Lemma 5.26 there exist $k_1, \ldots, k_m \in \mathbb{N}$ such that $\sum_{l=1}^m k_l = k$ and

$$N(k_1, \ldots, k_m) \geq c \cdot k^{(1-m)/2} \cdot p_1^{-k_1} \cdot \cdots \cdot p_m^{-k_m}$$

for some $c > 0$ independent of k. Then for every

$$\mathbf{i} \in \mathcal{J}_0 := \left\{(i_1, \ldots, i_k) \in \mathcal{I}^k : |\{j : 1 \leq j \leq k, i_j = l\}| = k_l \text{ for every } 1 \leq l \leq m\right\}$$

it follows that

$$\rho := r_{\mathbf{i}} = \prod_{l=1}^m r_l^{k_l}$$

and

$$|\mathcal{J}_0| = N(k_1, \ldots, k_m) \geq c \cdot k^{-m/2} \prod_{l=1}^m r_i^{-sk_l}.$$

Since $\{U_i\}_{i=1}^n$ is a finite $\frac{\varepsilon}{2}$-cover of $\overline{\mathcal{T}}$ we can find U_i such that for at least $n^{-1}|\mathcal{J}_0|$ words $\mathbf{i} \in \mathcal{J}_0$ we have that $T_{\mathbf{i}} \in U_i$. Let $\mathcal{J} = \{\mathbf{i} \in \mathcal{J}_0 : T_{\mathbf{i}} \in U_i\}$, then $\|T_{\mathbf{i}} - O_i\| < \frac{\varepsilon}{2}$, $r_{\mathbf{i}} = \rho$ for every $\mathbf{i} \in \mathcal{J}$ and

$$|\mathcal{J}| \geq n^{-1} N(k_1, \ldots, k_m) \geq n^{-1} c(\sqrt{k})^{-m} \prod_{l=1}^m r_i^{-sk_l}.$$

Let K be the attractor of the SS-IFS $\left\{\widehat{S}_{\mathbf{i}} \circ \widehat{S}_{\mathbf{j}_i} : \mathbf{i} \in \mathcal{J}\right\}$. Then $r := r_{\mathbf{i}} r_{\mathbf{j}_i} = \rho r_{\mathbf{j}_i}$ and $\left\|T_{\mathbf{i}} \circ T_{\mathbf{j}_i} - O\right\| < \varepsilon$ by (5.5) for every $\mathbf{i} \in \mathcal{J}$ and by (2.6) and Theorem 2.34

$$\begin{aligned}
\dim_H K = \frac{\log |\mathcal{J}|}{-\log \rho r_{\mathbf{j}_i}} &\geq \frac{-\log n + \log c - m \log(\sqrt{k}) - s\left(\sum_{l=1}^m k_l \log r_l\right)}{-\left(\sum_{l=1}^m k_l \log r_l\right) - \log r_{\mathbf{j}_i}} \\
&\geq s - \frac{\varepsilon}{2} > \dim_H K_j - \varepsilon
\end{aligned}$$

if k is large enough. $\square$

Lemma 5.28. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS in $\mathbb{R}^d$ that satisfies the SSC with attractor K and let $\varepsilon > 0$. Let $\widehat{K}$ be the attractor of the SS-IFS $\left\{S_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ where either $f_{\mathbf{i}} = S_1$ or $f_{\mathbf{i}} = Id_{\mathbb{R}^d}$. For k large enough $\dim_H K - \varepsilon < \dim_H \widehat{K} \leq \dim_H K$.*

Proof. It follows that $\dim_H \widehat{K} \leq \dim_H K$ because $\widehat{K} \subseteq K$. Since the SSC is satisfied $\dim_H K = s$ where $\sum_{i=1}^m r_i^s = 1$, see (2.6) and Theorem 2.34. Thus $\sum_{i=1}^m r_i^{s-\varepsilon} > 1$. Let $k \in \mathbb{N}$ such that $\left(\sum_{i=1}^m r_i^{s-\varepsilon}\right)^k > 1/r_1^{s-\varepsilon}$. Let $p_{\mathbf{i}} = r_1$ if $f_{\mathbf{i}} = S_1$ and $p_{\mathbf{i}} = 1$ if $f_{\mathbf{i}} = Id_{\mathbb{R}^d}$. Then

$$\sum_{\mathbf{i} \in \mathcal{I}^k} r_{\mathbf{i}}^{s-\varepsilon} p_{\mathbf{i}}^{s-\varepsilon} \geq \sum_{\mathbf{i} \in \mathcal{I}^k} r_{\mathbf{i}}^{s-\varepsilon} r_1^{s-\varepsilon} = \left(\sum_{i=1}^m r_i^{s-\varepsilon}\right)^k r_1^{s-\varepsilon} > 1$$

and hence $\dim_H K - \varepsilon = s - \varepsilon < \dim_H \widehat{K}$ because $\left\{S_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ satisfies the SSC. $\square$

Proof of Theorem 5.5. If $|\mathcal{T}_{j,G}| < \infty$ then Theorem 5.5 follows from Corollary 5.4 because $\mathcal{T}_{j,G} \cap \mathbb{SO}_2$ is a finite cyclic group since $\mathbb{SO}_2$ is commutative, so we assume that $|\mathcal{T}_{j,G}| = \infty$. By Theorem 5.1 there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_j$, $\dim_H K_j - \frac{\varepsilon}{2} < \dim_H \widehat{K}$ and the transformation group $\mathcal{T}$ of $\left\{\widehat{S}_i\right\}_{i=1}^m$ is dense in $\mathcal{T}_{j,G}$. It is easy to see that $|\mathcal{T}| = \infty$ implies that $\mathcal{T}$ contains a rotation of infinite order. If $\mathcal{T}$ contains reflections, without loss of generality say T_1 is a reflection, we iterate the SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ a large number of times and compose the orientation reversing maps with $\widehat{S}_1$. The new SS-IFS looks like $\left\{\widehat{S}_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ where $f_{\mathbf{i}} = \widehat{S}_1$ if $T_{\mathbf{i}} \notin \mathbb{SO}_2$ and $f_{\mathbf{i}} = Id_{\mathbb{R}^2}$ if $T_{\mathbf{i}} \in \mathbb{SO}_2$. Since $\mathcal{T}$ contains a rotation of infinite order it follows that the transformation group of $\left\{\widehat{S}_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ contains a rotation of infinite order. It follows from Lemma 5.28 that if we choose k large enough then the Hausdorff dimension of the attractor of $\left\{\widehat{S}_{\mathbf{i}} \circ f_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}^k\right\}$ approximates $\dim_H \widehat{K}$. Hence we can assume that there exists an SS-IFS $\left\{\widehat{S}_i\right\}_{i=1}^m$ that satisfies the SSC with attractor $\widehat{K}$ such that $\widehat{K} \subseteq K_j$, $\dim_H K_j - \frac{\varepsilon}{2} < \dim_H \widehat{K}$ and $\mathcal{T} \subseteq \mathbb{SO}_2$ contains a rotation of infinite order.

The rest of the proof is very similar to the proof of Theorem 5.3. There is no need for the $\frac{\varepsilon}{2}$-cover $\{U_l\}_{l=1}^p$ of $\overline{\mathcal{T}}$. Instead we fix $\mathbf{j} \in \mathcal{I}^l$ for some $l \in \mathbb{N}$ such that $T_{\mathbf{j}}$ is of infinite order. From here on we proceed as in the the proof of Theorem 5.3 with minor differences. Since $\mathbb{SO}_2$ is commutative it follows that for all the $N(k_1, \ldots, k_m)$ words $\mathbf{i} \in \mathcal{J}_0$ we have that $T_{\mathbf{i}} = T$ for some $T \in \mathcal{T}$. Then either T or $T \circ T_{\mathbf{j}}$ is of infinite order. Hence proceeding as in the proof of Theorem 5.3 we can show that either $\left\{\widehat{S}_{\mathbf{i}} : \mathbf{i} \in \mathcal{J}\right\}$ or $\left\{\widehat{S}_{\mathbf{i}} \circ \widehat{S}_{\mathbf{j}_l} : \mathbf{i} \in \mathcal{J}\right\}$ satisfies the theorem. $\square$

6 The equality of Hausdorff measure and Hausdorff content for subshifts of finite type and graph directed attractors

Hausdorff content is a concept closely related to Hausdorff measure, but perhaps less popular in the context of classical measure theory. That being said Hausdorff content enjoys greater regularity than Hausdorff measure and still gives Hausdorff dimension as the critical exponent. The main goal of this chapter is to understand further the relationship between Hausdorff measure and Hausdorff content in the context of subshifts of finite type and graph directed attractors. In particular, we are interested in when the Hausdorff measure and Hausdorff content of a set are equal at the Hausdorff dimension. The main result of this chapter shows that for a first level set of a subshift of finite type $\mathcal{H}^s_\infty\big(F^i_A\big) = \mathcal{H}^s\big(F^i_A\big)$ where $s = \dim_H F_A$. This implies that $\mathcal{H}^s_\infty\big(F\big) = \mathcal{H}^s\big(F\big)$ for graph directed self-similar sets too. We also investigate a packing measure analogue of this question. This chapter is based on a collaboration with Fraser [26].

6.1 Hausdorff measure and Hausdorff content

For every $s > \dim_H F$, we have $\mathcal{H}^s_\infty(F) = \mathcal{H}^s_\delta(F) = \mathcal{H}^s(F) = 0$ for every $\delta > 0$ and for every $s < \dim_\mathrm{H} F$, we have $\mathcal{H}^s_\infty(F) \leq \mathcal{H}^s_\delta(F) \leq \mathcal{H}^s(F) = \infty$, again for every $\delta > 0$, with the final inequality strict if F is bounded ($\mathcal{H}^s_\delta(F)$ is finite for every δ for F bounded). The case when $s = \dim_H F$ is more subtle, and is the case of interest. Then $\mathcal{H}^s_\infty(F) \leq \mathcal{H}^s_\delta(F) \leq \mathcal{H}^s(F)$ and $\mathcal{H}^s(F)$ may be zero, positive and finite, or infinite, but $\mathcal{H}^s_\infty(F)$ must be finite if F is bounded. Moreover, $\mathcal{H}^s_\infty(F) = 0$ if and only if $\mathcal{H}^s(F) = 0$.

The goal of this section is to study situations where $\mathcal{H}^s_\infty(F) = \mathcal{H}^s(F)$ where $s = \dim_H F$. Sets with this property were studied by Foran [30], where they were called *s-straight sets*. There are many advantages to having this equality as Hausdorff content is more easily analysed. For example, the expression $\sum_{k=1}^\infty \operatorname{diam}(U_k)^s$ gives a genuine upper bound for $\mathcal{H}^s_\infty(F)$ for every cover $\{U_k\}_{k=1}^\infty$, and for every $s \geq 0$ the function $\mathcal{H}^s_\infty$ acting on the set of compact subsets of a compact metric space equipped with the Hausdorff metric is an upper semicontinuous function, and thus Baire 1, whereas $\mathcal{H}^s$ is only Baire 2, see [55]. Another consequence is that $\mathcal{H}^s_\delta(F) = \mathcal{H}^s(F)$ for all $\delta > 0$.

Theorem 6.1. *Let Σ_A be an irreducible subshift of finite type with attractor F_A and let $s = \dim_\mathrm{H} F_A$. For all $\mathbf{i} \in \mathcal{J}^*$ we have*

$$\mathcal{H}^s_\infty\big(F^{\mathbf{i}}_A\big) = \mathcal{H}^s\big(F^{\mathbf{i}}_A\big).$$

Moreover, we can extend this to unions of 1-cylinders in the 'same family'. For all $i \in \mathcal{J}$,

$$\mathcal{H}^s_\infty\Bigg(\bigcup_{j \in \mathcal{J}: A_{i,j}=1} F^j_A\Bigg) = \mathcal{H}^s\Bigg(\bigcup_{j \in \mathcal{J}: A_{i,j}=1} F^j_A\Bigg).$$

We will prove Theorem 6.1 in Section 6.6. It is natural to wonder if the equality is still satisfied for the full set F_A, and not just cylinders and unions of cylinders in the 'same family'. We give an example in Section 6.3 which shows that this is not true. Delaware [11] proved that any set with finite $\mathcal{H}^s$ measure is σs-straight, in that it can be decomposed as a countable union of s-straight sets. This proved a conjecture of Foran [30]. Theorem 6.1 can be viewed as a strengthening of this result in the special case of subshifts of finite type. In particular, we prove that for irreducible A the set F_A can be decomposed into a finite union of s-straight sets

$$F_A = \bigcup_{i \in \mathcal{J}} F_A^i.$$

Theorem 6.1 was shown for self-similar sets rather than attractors of subshifts of finite type by Bandt and Graf [2, Proposition 3] assuming the OSC is satisfied. This result was generalised in Proposition 3.5, see [25, Proposition 1.11], for self-similar sets without assuming any separation condition. One might initially wonder if $\mathcal{H}^{\dim_H F}(F) = 0$ always holds when F is a self-similar set which cannot be defined via a system which satisfies the open set condition, but this is false, see Example 4.33. Thus this result provides non-trivial information even when the open set condition is not satisfied. Recall Theorem 2.34 that $\mathcal{H}^s(F) = 0$ if F is a self-similar set defined via a system which does not satisfy the open set condition and s is the similarity dimension but, as Example 4.33 shows, one can obtain positive Hausdorff measure at the Hausdorff dimension if this is less than the similarity dimension, even if the open set condition cannot be satisfied. Due to the equivalence in Section 2.2.3 we can deduce $\mathcal{H}^s_\infty(F) = \mathcal{H}^s(F)$ for graph directed self-similar sets from the result for subshifts of finite type.

Corollary 6.2. *Let F be a graph-directed self-similar set and let $s = \dim_{\mathrm{H}} F$. Then, regardless of separation conditions, $\mathcal{H}^s_\infty(F) = \mathcal{H}^s(F)$.*

Corollary 6.2 follows from Theorem 6.1 and Proposition 2.50.

6.2 Ahlfors regularity

For a subshift of finite type Σ_A and $k \in \mathbb{N}$ let

$$\mathcal{J}_A^k = \left\{ \mathbf{i} \in \mathcal{J}^k : \exists \alpha \in \Sigma_A \, \text{such that} \, \alpha|_k = \mathbf{i} \right\}.$$

If $\mathbf{i} = (i_0, \ldots, i_{k-1}), \mathbf{j} = (j_0, \ldots, j_{l-1}) \in \mathcal{J}^*$ then we write $\mathbf{i} * \mathbf{j} = (i_0, \ldots, i_{k-1}, j_0, \ldots, j_{l-1}) \in \mathcal{J}^*$.

Lemma 6.3. *Let Σ_A be an irreducible subshift of finite type with attractor F_A. Then there exists $c > 0$ such that for every $x \in F_A$, $j \in \mathcal{J}$ and $r \in (0, \operatorname{diam}(F_A^j))$ there exists $\mathbf{i} \in \mathcal{J}^*$ such that $F_A^{\mathbf{i}*j} \subseteq B(x, r)$ and $cr \leq \operatorname{diam}(F_A^{\mathbf{i}*j}) < r$.*

The proof of Lemma 6.3 is similar to the proof of Lemma 3.8. We omit the details.

Our results yield the following corollary.

Corollary 6.4. *Let Σ_A be an irreducible subshift of finite type with attractor F_A and let $s = \dim_{\mathrm{H}} F_A$. Then $\mathcal{H}^s(F_A) > 0$ if and only if F_A is Ahlfors regular. Moreover, this extends to any cylinder, i.e., for all $\mathbf{i} \in \mathcal{J}^*$, $\mathcal{H}^s(F_A^{\mathbf{i}}) > 0$ if and only if $F_A^{\mathbf{i}}$ is Ahlfors regular.*

Proof. We will prove the result for F_A; the result for cylinders is similar and omitted. Fix $r \in (0, \operatorname{diam}(F_A)]$ and $x \in F_A$. Assume that $\mathcal{H}^s(F_A^{\mathbf{i}}) > 0$. The lower bound that $cr^s \leq \mathcal{H}^s\big(F_A \cap B(x,r)\big)$ follows by choosing a first level cylinder with positive measure and then finding a copy of this cylinder inside $F_A \cap B(x,r)$ with diameter comparable to r, see Lemma 6.3 and then applying the scaling property for Hausdorff measure.

For the upper bound,

$$\begin{aligned}
\mathcal{H}^s\big(F_A \cap B(x,r)\big) &\leq \sum_{i \in \mathcal{J}} \mathcal{H}^s\big(F_A^i \cap B(x,r)\big) \\
&= \sum_{i \in \mathcal{J}} \mathcal{H}^s_\infty\big(F_A^i \cap B(x,r)\big) \qquad \text{by Theorem 6.1 and Lemma 2.2} \\
&\leq \sum_{i \in \mathcal{J}} \operatorname{diam}\big(F_A^i \cap B(x,r)\big)^s \\
&\leq M(2r).^s
\end{aligned}$$

The converse is obvious. □

Observe that the above corollary also applies to any collection of cylinders in F_A and due to the equivalence in Section 2.2.3 to graph-directed self-similar sets.

Corollary 6.5. *Let F be a graph-directed self-similar set and let $s = \dim_{\mathrm{H}} F$. Then, regardless of separation conditions, $\mathcal{H}^s(F) > 0$ if and only if F is Ahlfors regular.*

6.3 Examples where $\mathcal{H}^s_\infty(F) < \mathcal{H}^s(F) < \infty$

In this section we give examples which show that equality of Hausdorff measure and Hausdorff content at the critical dimension is actually a rather special property. In particular, we give several examples falling into natural classes of set for which one might hope to be able to extend Theorem 6.1, but for which equality does not hold. A natural situation to consider is attractors of more general iterated function systems.

Two of the most standard and important generalisations of self-similar sets are *self-affine sets*, that are attractors of IFSs where the defining maps are affine maps on some Euclidean space, and *self-conformal sets*, that are attractors of IFSs where the defining maps are conformal. We note that similarities are both affine and conformal. It is evident that for any compact set $F \subset \mathbb{R}^d$ with Hausdorff dimension equal to 1, we have

$$\mathcal{H}^1_\infty(F) \leq \operatorname{diam}(F).$$

However, if F is connected and not contained in a straight line, then

$$\mathcal{H}^1(F) > \operatorname{diam}(F).$$

This phenomenon provides us with several simple counter examples.

Self-affine sets: It was shown in [3] that there exist self-affine curves C in the plane which are differentiable at all but countably many points. In particular, these curves can have finite length but not lie in a straight line (see [3, Example 10] and [43, Example 6.2]). Such sets have Hausdorff dimension 1 and by the above argument satisfy

$$0 < \mathcal{H}^1_\infty(C) < \mathcal{H}^1(C) < \infty.$$

Self-conformal sets: The upper half A of the unit circle in the complex plane is a self-conformal set and has

$$\mathcal{H}^1_\infty(A) = 2 < \pi = \mathcal{H}^1(A).$$

The maps in the defining IFS for A are $z \mapsto \sqrt{z}$ and $z \mapsto i\sqrt{z}$, defined on a suitable open domain containing A.

Julia sets: the unit circle S^1 is the Julia set for the complex map $z \mapsto z^2$ and satisfies

$$\mathcal{H}^1_\infty(S^1) = 2 < 2\pi = \mathcal{H}^1(S^1).$$

Sub-self-similar sets: *Sub-self-similar sets*, introduced by Falconer in [21], are compact sets F satisfying

$$F \subseteq \bigcup_{i \in \mathcal{I}} S_i(F)$$

for some SS-IFS. For any SS-IFS with the unit square as its attractor, the boundary of the unit square $Q = \partial[0,1]^2$ is a sub-self-similar set and satisfies

$$\mathcal{H}^1_\infty(Q) = \sqrt{2} < 4 = \mathcal{H}^1(Q).$$

All of our counter examples in these classes were using sets with dimension 1. Could there be different phenomena at work for non-integral dimensions? Note that we cannot give a simple condition guaranteeing $\mathcal{H}^s_\infty(F) < \mathcal{H}^s(F)$ apart from for connected sets F not lying in a straight line with Hausdorff dimension $s = 1$. This is because such sets may be s-straight by the result of Delaware mentioned previously [11].

Finally we give two simple examples which show that Theorem 6.1 is sharp, in some sense.

Non-irreducible subshift of finite type: Consider the subshift of finite type on the alphabet $\{0, 1, 2\}$ given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and associate any iterated function system consisting of three similarities on $[0,1]$ which map $[0,1]$ to three disjoint intervals. Here A is not irreducible and so does not fall into the class considered by Theorem 6.1. The limit set $F_A = \Pi\big(\Sigma_A\big)$ consists of only four points and so F and all of its children have Hausdorff dimension 0, but nevertheless

$$\mathcal{H}^0_\infty\big(F_A^2\big) = 1 < 2 = \mathcal{H}^0\big(F_A^2\big).$$

Full set for irreducible and aperiodic subshift of finite type: Now we will show that one cannot hope to have $\mathcal{H}^s_\infty(F_A) = \mathcal{H}^s(F_A)$ for even an aperiodic subshift of finite type (which we recall is a stronger condition than irreducible) that satisfies the OSC. Consider the alphabet $\{0, 1, 2, 3\}$ and let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

which is quickly seen to be aperiodic. Define similarities on the unit square by

$$S_0(x, y) = (x/2, y/2), \qquad S_1(x, y) = (-x/2, y/2) + (1/2, 1/2),$$

$$S_2(x, y) = (x/2, y/2) + (1/2, 0), \quad \text{and} \quad S_3(x, y) = (-x/2, y/2) + (1, 1/2).$$

It is easy to see that

$$F_A = (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])$$

which satisfies

$$\mathcal{H}^1_\infty(F_A) = \sqrt{2} < 2 = \mathcal{H}^1(F_A).$$

Of course Theorem 6.1 still correctly states that

$$\mathcal{H}^1_\infty\big(F_A^0 \cup F_A^1\big) = \mathcal{H}^1\big(F_A^0 \cup F_A^1\big) \quad \text{and} \quad \mathcal{H}^1_\infty\big(F_A^2 \cup F_A^3\big) = \mathcal{H}^1\big(F_A^2 \cup F_A^3\big),$$

noting that

$$F_A^0 \cup F_A^1 = \{0\} \times [0, 1] \quad \text{and} \quad F_A^2 \cup F_A^3 = \{1\} \times [0, 1].$$

Self-affine sets with infinite Hausdorff measure: For our earlier examples $\mathcal{H}^s_\infty(F) < \mathcal{H}^s(F) < \infty$ holds where $s = \dim_H F$. If $\mathcal{H}^s(F) = \infty$ then obviously the Hausdorff content and Hausdorff measure differs since for compact sets $\mathcal{H}^s_\infty(F) < \infty$. Peres [63] proved that for every member F of a certain family of self-affine sets $\mathcal{H}^s(F) = \infty$ where $s = \dim_H F$.

6.4 The question of packing measure

In this section we address the question of whether analogous results can be obtained for packing measure, see Section 2.1.3, and a suitably defined 'packing content'.

Proposition 6.6. *Let F be a compact subset of $\mathbb{R}^d$ with the property that for every open ball B centred in F, there exists a bi-Lipschitz map S on $\mathbb{R}^d$ such that $S(F) \subseteq B \cap F$. Then for all $s \geq 0$ we have*

$$\mathcal{P}^s(F) = \mathcal{P}^s_0(F).$$

Proof. For any compact set $F \subset \mathbb{R}^d$, if $\mathcal{P}^s_0(F) < \infty$, then $\mathcal{P}^s(F) = \mathcal{P}^s_0(F)$, by the main result in [29]. In the case when $\mathcal{P}^s_0(F) = \infty$, the additional assumption implies that $\mathcal{P}^s_0(B \cap F) = \infty$ for all open balls intersecting F, which by [36, Lemma 4], implies that $\mathcal{P}^s(F) = \infty$. □

For this reason we can concern ourselves only with the packing pre-measure, which is easier to understand. The first question is, how should we define the packing (pre) content? If we naively define it by just removing the bounds on the diameters of the balls in the packing, then the answer is always infinity, as long as $s > 0$ and $F \neq \emptyset$. This is because one can just take a packing by a single ball with unbounded diameter. Possible alternatives would be either to insist that there are at least two balls in every packing, or to bound the radii by something concrete, such as the diameter of F itself. However, it might be more natural to try to prove that for sufficiently small δ, the equality $\mathcal{P}_0^s(F) = \mathcal{P}_\delta^s(F)$ is satisfied. We adopt this third approach. It is natural to ask, do we expect this to be true in the same setting as Theorem 6.1?

"*If F is self-similar, then does there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have*

$$\mathcal{P}^s(F) = \mathcal{P}_0^s(F) = \mathcal{P}_\delta^s(F)?\text{''}$$

One strange consequence of this would be that for such sets the packing measure is always strictly positive. In the same way that $\mathcal{H}_\delta^s(F)$ is always finite for bounded sets, we have that $\mathcal{P}_\delta^s(F)$ is always positive for arbitrary non-empty sets. Interestingly it was an important question for about 15 years whether or not it was possible for a self-similar set in the line to have zero packing measure in its dimension if $\dim_P F < 1$, see [65, Question 2.3], but this was recently resolved by Orponen [62], who provided a family of self-similar sets for whose elements F (of course not satisfying the open set condition) $\mathcal{P}^{\dim_P F}(F) = 0$. Thus the answer to the above question is immediately 'no'. However, we proved positive answer in the case when the SSC is satisfied.

Theorem 6.7. *Let $\{S_i\}_{i=1}^m$ be an SS-IFS that satisfies the SSC, with attractor $F \subseteq \mathbb{R}^d$ and let $s = \dim_{\mathrm{P}} F$. Then, there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have*

$$0 < \mathcal{P}^s(F) = \mathcal{P}_0^s(F) = \mathcal{P}_\delta^s(F) < \infty.$$

We will prove Theorem 6.7 in Section 6.7. By the above discussion, this result does not extend to F which do not satisfy the open set condition. It is also easy to see that it does not extend to the open set condition case either. For example, the unit interval I is a self-similar set that can be defined via an SS-IFS satisfying the OSC but not the SSC. Elementary calculations yield that $\mathcal{P}^1(I) = 1$, but that $\mathcal{P}_\delta^1(I) = 1 + \delta$ for all δ. We pose the question of whether the appropriate converse of Theorem 6.7 is true.

Question 6.8. *Does there exists a self-similar set F defined via a system satisfying the OSC, but for which there is no SS-IFS satisfying the SSC with F as the attractor, for which there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have*

$$0 < \mathcal{P}^s(F) = \mathcal{P}_0^s(F) = \mathcal{P}_\delta^s(F) < \infty?$$

We generalise Theorem 6.7 for graph-directed self-similar sets and subshifts of finite type.

Theorem 6.9. *Let $\{S_e : e \in \mathcal{E}\}$ be a strongly connected GD-IFS that satisfies the SSC, with attractor $(K_1, \ldots, K_q)$ and let s be the common packing dimension of the sets $\{K_i\}_{i\in\mathcal{V}}$. Then, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and all $i \in \mathcal{V}$ we have*

$$0 < \mathcal{P}^s(K_i) = \mathcal{P}_0^s(K_i) = \mathcal{P}_\delta^s(K_i) < \infty.$$

We will prove Theorem 6.9 in Section 6.7. Due to Proposition 2.49 it follows that this result generalises to attractors of subshift of finite types.

6.5 A Vitali like exhaustion lemma for subshifts of finite type

In this section we prove an exhaustion lemma for Hausdorff measure, similar to Proposition 3.3, which may be of interest in its own right. It shows that we can exhaust the Hausdorff measure of a (potentially overlapping) subshift of finite type by infinitely many, disjoint, images of first level cylinders.

Let

$$\mathcal{J}_A^* = \{\mathbf{i} \in \mathcal{J}^* : \exists \alpha \in \Sigma_A \,\text{and}\, k \in \mathbb{N} \,\text{such that}\, \alpha|_k = \mathbf{i}\}$$

and for $\mathbf{i} \in \mathcal{J}_A^*$ let

$$\mathcal{J}_A^{\mathbf{i}*} = \left\{\mathbf{j} \in \mathcal{J}_A^* : \mathbf{j}|_{|\mathbf{i}|} = \mathbf{i}\right\}.$$

For $\mathbf{i} = (i_0, \ldots, i_{k-1}) \in \mathcal{J}^*$ with $|\mathbf{i}| \geq 1$ we define $(\mathbf{i})_0 = i_0$ and $(\mathbf{i})_{last} = i_{k-1}$ and $\tau(\mathbf{i}) = (i_0, \ldots, i_{k-2})$. If $\mathbf{i} = (i_0, \ldots, i_{k-1}), \mathbf{j} = (j_0, \ldots, j_{l-1}) \in \mathcal{J}_A^*$ are such that $A_{(\mathbf{i})_{last},(\mathbf{j})_0} = 1$ then $\mathbf{i} * \mathbf{j} = (i_0, \ldots, i_{k-1}, j_0, \ldots, j_{l-1}) \in \mathcal{J}_A^*$.

Lemma 6.10. *Let Σ_A be an irreducible subshift of finite type with attractor F_A, let $s = \dim_{\mathrm{H}} F_A$ and assume that $\mathcal{H}^s(F_A) > 0$. Then for each $j \in \mathcal{J}$, there exists a collection $\mathcal{I}_\infty^j$ of finite words $\mathbf{i} \in \mathcal{J}^*$ that satisfies the following properties:*
(i) the first symbol is j, i.e. $(\mathbf{i})_0 = j$,
(ii) the last symbol is j, i.e. $(\mathbf{i})_{last} = j$,
(iii) there exists $\alpha \in \Sigma_A$ and $k \in \mathbb{N}$ such that $\alpha|_k = \mathbf{i}$ or, in other words, $\mathbf{i} \in \mathcal{J}_A^*$,
(iv) for $\mathbf{i}, \mathbf{j} \in \mathcal{I}_\infty^j$ with $\mathbf{i} \neq \mathbf{j}$ we have that

$$F_A^{\mathbf{i}} \cap F_A^{\mathbf{j}} = \emptyset,$$

(v)

$$\mathcal{H}^s\left(F_A^j \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty^j} F_A^{\mathbf{i}}\right)\right) = 0,$$

(vi) the contraction ratios satisfy a Hutchinson-Moran type expression for Hausdorff dimension, i.e.

$$\sum_{\mathbf{i}\in\mathcal{I}_\infty^j} r_{\tau(\mathbf{i})}^s = 1.$$

Proof. Since A is irreducible, for every $i \in \mathcal{J} \setminus \{j\}$ we can find $\mathbf{i}_i \in \mathcal{J}_A^{i*}$ such that $(\mathbf{i}_i)_0 = i$ and $(\mathbf{i}_i)_{last} = j$. Thus if $\mathbf{i} \in \mathcal{J}_A^{j*}$ and $(\mathbf{i})_{last} = i$ then $\tau(\mathbf{i}) * \mathbf{i}_i \in \mathcal{J}_A^{j*}$ and $(\tau(\mathbf{i}) * \mathbf{i}_i)_{last} = j$. Let

$$r_{\min} = \min \left\{ \frac{\mathcal{H}^s\left(F_A^{\mathbf{i}_i}\right)}{\mathcal{H}^s\left(F_A^{i}\right)} : i \in \mathcal{J} \setminus \{j\} \right\} \in (0,1). \tag{6.1}$$

We define a sequence $\mathcal{I}_0^j, \mathcal{I}_1^j, \ldots$ inductively where $\mathcal{I}_n^j$ satisfies properties *(i), (iii), (iv)* and *(v)*. The collection of sets $\left\{F_A^{\mathbf{i}} : \mathbf{i} \in \mathcal{J}_A^{j*}, |\mathbf{i}| \geq 2\right\}$ is a Vitali cover of F_A^j and hence by Proposition 2.18 there exists $\mathcal{I}_0^j \subseteq \left\{\mathbf{i} \in \mathcal{J}_A^{j*} : |\mathbf{i}| \geq 2\right\}$ such that $F_A^{\mathbf{i}} \cap F_A^{\mathbf{j}} = \emptyset$ for $\mathbf{i}, \mathbf{j} \in \mathcal{I}_0^j$, $\mathbf{i} \neq \mathbf{j}$ and

$$\mathcal{H}^s\left(F_A^j \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{I}_0^j} F_A^{\mathbf{i}}\right)\right) = 0.$$

Once $\mathcal{I}_n^j$ is defined we define $\mathcal{I}_{n+1}^j$ as follows. First, for each $\mathbf{i} \in \mathcal{I}_n^j$ we define a set $\mathcal{I}_{n+1,\mathbf{i}}^j$. If $(\mathbf{i})_{last} = j$ then $\mathcal{I}_{n+1,\mathbf{i}}^j = \{\mathbf{i}\}$. If $(\mathbf{i})_{last} = i \neq j$ then

$$\left\{F_A^{\tau(\mathbf{i}) * \mathbf{j}} : \mathbf{j} \in \mathcal{J}_A^{i*}, F_A^{\tau(\mathbf{i}) * \mathbf{j}} \cap F_A^{\tau(\mathbf{i}) * \mathbf{i}_i} = \emptyset\right\}$$

is a Vitali cover of $F_A^{\mathbf{i}} \setminus F_A^{\tau(\mathbf{i}) * \mathbf{i}_i}$ and hence by Proposition 2.18 there exists

$$\mathcal{J}_{n+1,\mathbf{i}} \subseteq \left\{\mathbf{j} : \mathbf{j} \in \mathcal{J}_A^{i*}, F_A^{\tau(\mathbf{i}) * \mathbf{j}} \cap F_A^{\tau(\mathbf{i}) * \mathbf{i}_i} = \emptyset\right\}$$

such that $F_A^{\tau(\mathbf{i}) * \mathbf{j}_1} \cap F_A^{\tau(\mathbf{i}) * \mathbf{j}_2} = \emptyset$ for all $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{J}_{n+1,\mathbf{i}}^j$, with $\mathbf{j}_1 \neq \mathbf{j}_2$, and

$$\mathcal{H}^s\left(\left(F_A^{\mathbf{i}} \setminus F_A^{\tau(\mathbf{i}) * \mathbf{i}_i}\right) \setminus \left(\bigcup_{\mathbf{j} \in \mathcal{J}_{n+1,\mathbf{i}}^j} F_A^{\tau(\mathbf{i}) * \mathbf{j}}\right)\right) = 0.$$

Now let

$$\mathcal{I}_{n+1,\mathbf{i}}^j = \{\tau(\mathbf{i}) * \mathbf{i}_i\} \bigcup \left\{\tau(\mathbf{i}) * \mathbf{j} : \mathbf{j} \in \mathcal{J}_{n+1,\mathbf{i}}^j\right\}$$

and

$$\mathcal{I}_{n+1}^j = \bigcup_{\mathbf{i} \in \mathcal{I}_n} \mathcal{I}_{n+1,\mathbf{i}}^j.$$

Finally we define

$$\mathcal{I}_\infty^j = \bigcap_{n_1=1}^{\infty} \bigcup_{n_2=n_1}^{\infty} \mathcal{I}_{n_2}^j.$$

Clearly $F_A^{\mathbf{i}} \cap F_A^{\mathbf{j}} = \emptyset$ for $\mathbf{i}, \mathbf{j} \in \mathcal{I}_\infty^j$, $\mathbf{i} \neq \mathbf{j}$. If $\mathbf{i} \in \mathcal{I}_n^j$ and $(\mathbf{i})_{last} \neq j$ then $\mathbf{i} \notin \mathcal{I}_{n+l}^j$ for every positive integer l, hence $\mathbf{i} \notin \mathcal{I}_\infty^j$. So $(\mathbf{i})_{last} = j$ for all $\boldsymbol{i} \in \mathcal{I}_\infty^j$. Clearly

$$\mathcal{H}^s\left(F_A^j \setminus \left(\bigcup_{\boldsymbol{i} \in \mathcal{I}_n^j} F_A^{\boldsymbol{i}}\right)\right) = 0 \tag{6.2}$$

for every positive integer n. For $\mathbf{i} \in \mathcal{I}_n^j$ such that $(\mathbf{i})_{last} = i \neq j$ we have that

$$\left\{\mathbf{j} : \tau(\mathbf{i}) * \mathbf{j} \in \mathcal{I}_{n+1}^j, (\tau(\mathbf{i}) * \mathbf{j})_{last} \neq j\right\} \subseteq \mathcal{J}_{n+1,\mathbf{i}}^j$$

and

$$\begin{aligned}\mathcal{H}^s\left(F_A^{\tau(\mathbf{i})*\mathbf{i}_i}\right) = \mathcal{H}^s\left(S_{\tau(\mathbf{i})*\tau(\mathbf{i}_i)}(F_A^j)\right) &= r_{\tau(\mathbf{i})}^s r_{\tau(\mathbf{i}_i)}^s \mathcal{H}^s\left(F_A^j\right)\frac{\mathcal{H}^s\left(F_A^i\right)}{\mathcal{H}^s\left(F_A^i\right)} \\ &= \mathcal{H}^s\left(F_A^{\mathbf{i}_i}\right)\frac{\mathcal{H}^s\left(F_A^{\mathbf{i}}\right)}{\mathcal{H}^s\left(F_A^i\right)} \\ &\geq r_{\min}\mathcal{H}^s\left(F_A^{\mathbf{i}}\right) \qquad (6.3)\end{aligned}$$

by (6.1). Also $(\tau(\mathbf{i}) * \mathbf{i}_i)_{last} = j$ by definition. Therefore

$$\mathcal{I}_{n+1}^j \setminus \mathcal{I}_\infty^j \subseteq \bigcup_{\mathbf{i}\in\mathcal{I}_n^j\setminus\mathcal{I}_\infty^j} \left\{\tau(\mathbf{i}) * \mathbf{j} : \mathbf{j} \in \mathcal{J}_{n+1,\mathbf{i}}^j\right\}$$

and

$$\begin{aligned}\mathcal{H}^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}_{n+1}^j\setminus\mathcal{I}_\infty^j} F_A^{\mathbf{i}}\right) &\leq \sum_{\mathbf{i}\in\mathcal{I}_n^j\setminus\mathcal{I}_\infty^j} \mathcal{H}^s\left(\bigcup_{\mathbf{j}\in\mathcal{J}_{n+1,\mathbf{i}}^j} F_A^{\tau(\mathbf{i})*\mathbf{j}}\right) \\ &\leq \sum_{\mathbf{i}\in\mathcal{I}_n^j\setminus\mathcal{I}_\infty^j} \mathcal{H}^s\left(F_A^{\mathbf{i}} \setminus F_A^{\tau(\mathbf{i})*\mathbf{i}_i}\right) \\ &\leq \sum_{\mathbf{i}\in\mathcal{I}_n^j\setminus\mathcal{I}_\infty^j} \left(\mathcal{H}^s\left(F_A^{\mathbf{i}}\right) - r_{\min}\mathcal{H}^s\left(F_A^{\mathbf{i}}\right)\right) \qquad \text{by (6.3)} \\ &= \sum_{\mathbf{i}\in\mathcal{I}_n^j\setminus\mathcal{I}_\infty^j} (1 - r_{\min}) \cdot \mathcal{H}^s\left(F_A^{\mathbf{i}}\right) \\ &= (1 - r_{\min}) \cdot \mathcal{H}^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}_n^j\setminus\mathcal{I}_\infty^j} F_A^{\mathbf{i}}\right).\end{aligned}$$

Hence

$$\mathcal{H}^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}_{n+1}^j\setminus\mathcal{I}_\infty^j} F_A^{\mathbf{i}}\right) \leq (1 - r_{\min})^n \cdot \mathcal{H}^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}_0^j\setminus\mathcal{I}_\infty^j} F_A^{\mathbf{i}}\right) \leq (1 - r_{\min})^n \cdot \mathcal{H}^s\left(F_A^j\right)$$

for all $n \in \mathbb{N}$ and combined with (6.2) we get that

$$\mathcal{H}^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}_{n+1}^j\cap\mathcal{I}_\infty^j} F_A^{\mathbf{i}}\right) \geq \left(1 - (1 - r_{\min})^n\right) \cdot \mathcal{H}^s\left(F_A^j\right).$$

Thus

$$\mathcal{H}^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}^j_\infty} F_A^{\mathbf{i}}\right) \geq \mathcal{H}^s\left(F_A^j\right)$$

and so

$$\mathcal{H}^s\left(F_A^j \setminus \left(\bigcup_{\mathbf{i}\in\mathcal{I}^j_\infty} F_A^{\mathbf{i}}\right)\right) = 0.$$

Thus the collection $\mathcal{I}^j_\infty$ satisfies properties *(i)-(v)*. Property *(vi)* follows easily from *(iv)* and *(v)* since

$$\mathcal{H}^s\left(F_A^j\right) = \sum_{\mathbf{i}\in\mathcal{I}^j_\infty} \mathcal{H}^s\left(F_A^{\mathbf{i}}\right) = \sum_{\mathbf{i}\in\mathcal{I}^j_\infty} \mathcal{H}^s\left(S_{\tau(\mathbf{i})}\left(F_A^j\right)\right) = \sum_{\mathbf{i}\in\mathcal{I}^j_\infty} r^s_{\tau(\mathbf{i})}\mathcal{H}^s\left(F_A^j\right)$$

and the fact that we can divide by $\mathcal{H}^s\left(F_A^j\right) > 0$. □

6.6 Proof of Theorem 6.1

In this section we will prove Theorem 6.1. It is trivially true if $\mathcal{H}^s(F_A) = 0$, so we assume otherwise. Fix $i \in \mathcal{J}$ and $\varepsilon > 0$. Choose a countable open cover $\{U_k\}_{k\in\mathcal{K}}$ of F_A^i which satisfies

$$\sum_{k\in\mathcal{K}} \operatorname{diam}(U_k)^s \leq \mathcal{H}^s_\infty(F_A^i) + \varepsilon. \tag{6.4}$$

Since F_A^i is bounded we can assume that there is a uniform bound on the diameters of the U_k. Let $\mathcal{I}^i_\infty$ be the 'exhausting set' from Lemma 6.10. For $n \in \mathbb{N}$, let

$$\mathcal{I}^{i,n}_\infty = \left\{\mathbf{i}' \in \mathcal{J}^* : \mathbf{i}' = \tau(\mathbf{i}^0)\tau(\mathbf{i}^1)\dots\tau(\mathbf{i}^{n-1}) \text{ where } \mathbf{i}^l \in \mathcal{I}^i_\infty \text{ for } l = 0,\dots,n-1\right\}. \tag{6.5}$$

By properties *(i)* and *(ii)* in Lemma 6.10 the set $\mathcal{I}^{i,n}_\infty$ is a set of restricted words from Σ^i_A. Moreover, it follows from property *(v)* in Lemma 6.10 that, for all $n \in \mathbb{N}$,

$$\mathcal{H}^s\left(F_A^i \setminus \bigcup_{\mathbf{i}\in\mathcal{I}^{i,n}_\infty} S_{\mathbf{i}}(F_A^i)\right) = 0. \tag{6.6}$$

Observe that, for each $n \in \mathbb{N}$,

$$\{S_{\mathbf{i}}(U_k)\}_{\mathbf{i}\in\mathcal{I}^{i,n}_\infty, k\in\mathcal{K}}$$

is a cover of $\bigcup_{\mathbf{i}\in\mathcal{I}^{i,n}_\infty} S_{\mathbf{i}}(F_A^i)$. Let $\delta > 0$ and choose $n \in \mathbb{N}$ sufficiently large to ensure that

$$\sup_{\mathbf{i}\in\mathcal{I}^{i,n}_\infty, k\in\mathcal{K}} \operatorname{diam}\big(S_{\mathbf{i}}(U_k)\big) \leq \delta$$

and thus

$$\{S_{\mathbf{i}}(U_k)\}_{\mathbf{i}\in\mathcal{I}^{i,n}_\infty, k\in\mathcal{K}}$$

is a countable open δ-cover of $\bigcup_{\mathbf{i}\in\mathcal{I}_\infty^{i,n}} S_{\mathbf{i}}(F_A^i)$. It follows that

$$\begin{aligned}
\mathcal{H}_\delta^s(F_A^i) &\leq \mathcal{H}_\delta^s\left(\bigcup_{\mathbf{i}\in\mathcal{I}_\infty^{i,n}} S_{\mathbf{i}}(F_A^i)\right) + \mathcal{H}_\delta^s\left(F_A^i \setminus \bigcup_{\mathbf{i}\in\mathcal{I}_\infty^{i,n}} S_{\mathbf{i}}(F_A^i)\right) \\
&\leq \sum_{k\in\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{I}_\infty^{i,n}} \operatorname{diam}\big(S_{\mathbf{i}}(U_k)\big)^s \qquad \text{by (6.6)} \\
&\leq \sum_{k\in\mathcal{K}} \operatorname{diam}(U_k)^s \sum_{\mathbf{i}\in\mathcal{I}_\infty^{i,n}} r_{\mathbf{i}}^s \\
&\leq \Big(\mathcal{H}_\infty^s(F_A^i) + \varepsilon\Big) \cdot \left(\sum_{\mathbf{j}\in\mathcal{I}_\infty^i} r_{\tau(\mathbf{j})}^s\right)^n \qquad \text{by (6.4) and (6.5)} \\
&= \mathcal{H}_\infty^s(F_A^i) + \varepsilon
\end{aligned}$$

by property *(vi)* from Lemma 6.10. Taking the limit as $\delta \to 0$ and noting that $\varepsilon > 0$ was arbitrary, yields $\mathcal{H}^s(F_A^i) \leq \mathcal{H}_\infty^s(F_A^i)$. The reverse inequality is always satisfied by (2.1).

The final part of Theorem 6.1 follows by a simple trick. Let $i \in \mathcal{J}$ and observe that

$$F_A^i = S_i\left(\bigcup_{j\in\mathcal{J}:A_{i,j}=1} F_A^j\right)$$

and so

$$r_i^s \cdot \mathcal{H}_\infty^s\left(\bigcup_{j\in\mathcal{J}:A_{i,j}=1} F_A^j\right) = \mathcal{H}_\infty^s(F_A^i) = \mathcal{H}^s(F_A^i) = r_i^s \cdot \mathcal{H}^s\left(\bigcup_{j\in\mathcal{J}:A_{i,j}=1} F_A^j\right)$$

where the middle equality is due to the first part of the theorem. Dividing by r_i^s completes the proof. □

6.7 Proof of Theorem 6.7 and Theorem 6.9

Proof of Theorem 6.7. The SS-IFS $\{S_i\}_{i\in\mathcal{I}}$ that satisfies the SSC, this implies that we can find a bounded open set $\mathcal{O} \subseteq \mathbb{R}^d$ such that $F \subseteq \mathcal{O}$ and $\bigcup_{i\in\mathcal{I}} S_i(\mathcal{O}) \subseteq \mathcal{O}$ is a disjoint union. Let

$$\delta_0 = \frac{1}{2} \cdot \inf_{x\in F} \inf_{y\in\mathbb{R}^d\setminus\mathcal{O}} \|x - y\|$$

which is strictly positive since F is closed.

First assume that $\mathcal{P}_\delta^s(F) < \infty$ for every $\delta \in (0, \delta_0)$. Later we will see that $\mathcal{P}_\delta^s(F) = \infty$ is impossible for $\delta \in (0, \delta_0)$. Let $\varepsilon > 0$, let $\delta \in (0, \delta_0)$ and let $\{B_k\}_{k\in\mathcal{K}}$ be a countable collection of disjoint closed balls centred in F with diameter less than or equal to δ which satisfies

$$\sum_{k\in\mathcal{K}} \operatorname{diam}(B_k)^s \geq \mathcal{P}_\delta^s(F) - \varepsilon. \tag{6.7}$$

Since $B_k \subset \mathcal{O}$ for all $k \in \mathcal{K}$ and by the choice of $\mathcal{O}$, the collection

$$\{S_{\mathbf{i}}(B_k)\}_{\mathbf{i}\in\mathcal{I}^n, k\in\mathcal{K}}$$

is a countable collection of disjoint closed balls centred in F. Let $\eta \in (0,\delta)$ and choose $n \in \mathbb{N}$ so large that

$$\sup_{\mathbf{i}\in\mathcal{I}^n, k\in\mathcal{K}} \operatorname{diam}\big(S_{\mathbf{i}}(B_k)\big) < \eta.$$

It follows that

$$\begin{aligned}
\mathcal{P}^s_\eta(F) &\geq \sum_{k\in\mathcal{K}}\sum_{\mathbf{i}\in\mathcal{I}^n} \operatorname{diam}\big(S_{\mathbf{i}}(B_k)\big)^s \\
&= \sum_{k\in\mathcal{K}} \operatorname{diam}(B_k)^s \sum_{\mathbf{i}\in\mathcal{I}^n} r_{\mathbf{i}}^s \\
&\geq \Big(\mathcal{P}^s_\delta(F) - \varepsilon\Big) \cdot \left(\sum_{i\in\mathcal{I}} r_i^s\right)^n \qquad \text{by (6.7)} \\
&= \mathcal{P}^s_\delta(F) - \varepsilon
\end{aligned}$$

by the Hutchinson-Moran formula for packing dimension, see (2.6) and Proposition 2.33. Taking the limit as $\eta \to 0$ and noting that $\varepsilon > 0$ was arbitrary, yields $\mathcal{P}^s(F) = \mathcal{P}^s_0(F) \geq \mathcal{P}^s_\delta(F)$. The reverse inequality is always satisfied by (2.3), which completes the proof.

Now assume that $\mathcal{P}^s_\delta(F) = \infty$ for some $\delta \in (0,\delta_0)$. Via a similar argument to the one above, this implies that $\mathcal{P}^s_\eta(F) > K$ for every $K > 0$ and every $\eta \in (0,\delta)$, hence $\mathcal{P}^s_0(F) = \infty$ but this is a contradiction since every self-similar set has finite packing measure (and pre-measure) in the packing dimension, see [15, Exercise 3.2]. □

The reason this proof cannot be extended to the open set condition case is because in that case the number δ_0 may be zero and iterations of packings may no longer be packings. This is one of the reasons packings are sometimes more difficult to control than covers. The proof of Theorem 6.9 is similar and we just provide a sketch. First we prove a simple lemma. Recall that we say $v \leq v'$ for vectors $v, v' \in \mathbb{R}^q$ if each entry in v is less than or equal to the corresponding entry in v'. We say that v is non-negative if $0 \leq v$. Similar notations apply to matrices.

Lemma 6.11. *Let A be a non-negative irreducible matrix of spectral radius 1 and x be a non-negative vector such that $A^m x \leq x$ for large enough m. Then $Ax = x$.*

Proof. Observe that A^m is also an irreducible matrix with spectral radius 1. Hence $A^m x = x$ by [5, Theorem 1.3.28], therefore $Ax = x$ by Theorem 2.44. □

Proof of Theorem 6.9. Let s be the similarity dimension of the GD-IFS $\{S_e : e \in \mathcal{E}\}$ and let $u^\intercal = (\mathcal{P}^s(K_1), ..., \mathcal{P}^s(K_q))$. Since $G(\mathcal{V},\mathcal{E})$ is strongly connected A^s is irreducible. Furthermore, the SSC is satisfied, so $0 < \mathcal{P}^s(K_i) < \infty$ for every $i \in \mathcal{V}$ and $A^s u = u$ (by a similar argument to Remark 2.45). Let $u_\delta^\intercal = (\mathcal{P}^s_\delta(K_1), ..., \mathcal{P}^s_\delta(K_q))$. Since the strong separation condition is satisfied there exists a collection of open sets $\{\mathcal{O}_i\}_{i\in\mathcal{V}}$ such that $K_i \subseteq \mathcal{O}_i$ and

$$\bigcup_{j\in\mathcal{V}} \bigcup_{e\in\mathcal{E}_{i,j}} S_e(\mathcal{O}_j) \subseteq \mathcal{O}_i$$

is a disjoint union for every $i \in \mathcal{V}$. Let

$$\delta_0 = \frac{1}{2} \cdot \min_{i\in\mathcal{V}} \inf_{x\in K_i} \inf_{y\in\mathbb{R}^d\setminus\mathcal{O}_i} \|x-y\|.$$

A similar argument to the proof of Theorem 6.7 shows that for large enough n depending on η we have that

$$(A^s)^n u_\delta \leq u_\eta \leq u_\delta \tag{6.8}$$

for $\delta \in (0, \delta_0)$ and $0 < \eta < \delta$. It follows by Lemma 6.11 that equality holds in (6.8). Hence $u_\delta = u_\eta = u$ for $0 < \eta < \delta < \delta_0$. □

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